Efficient Delta-decision Procedure

[Thesis Proposal]

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Chapter 1
Introduction

Decision Problems over the Reals

Given an arbitrary first-order sentence over $\langle \mathbb{R}, \geq, \mathcal{F} \rangle$, such as

$$\varphi = Q_1^{[l_1,u_1]} x_1 \dots Q_n^{[l_n,u_n]} x_n . \bigwedge_i \left(\bigvee_j f_{i,j}(\vec{x}) > 0 \vee \bigvee_k f_{i,k}(\vec{x}) \geq 0 \right)$$

where $f \in \mathcal{F}$, can we compute whether φ is true or false?

- Complexity results of **non-linear** arithmetic over the **Reals**
  - **Decidable** if φ only contains polynomials [Tarski51]
  - **Undecidable** if φ includes trigonometric functions (i.e. sin)
- **Real-world problems** contain **complex nonlinear functions** (trigonometric functions, log, exp, ODEs)

Delta-decision Problem

- Given a first-order formula over the Real φ, and a positive rational number δ, **delta-decision problem** asks for one of the following answers:

  - **UNSAT**: φ is unsatisfiable
  - **δ-SAT** : $\varphi^{-\delta}$ is satisfiable.

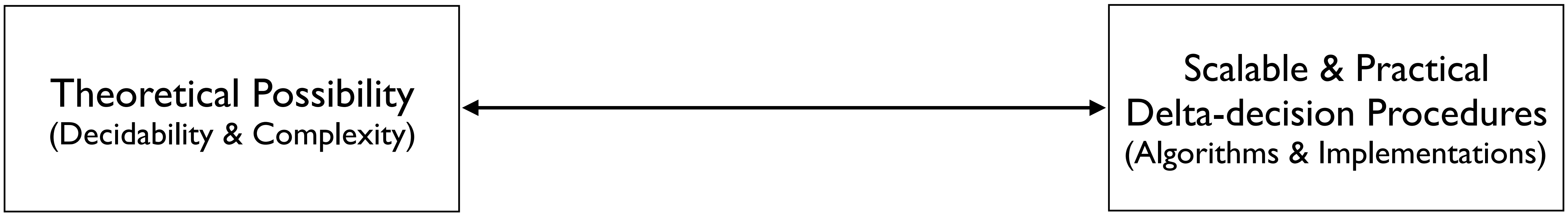
  where $\varphi^{-\delta}$is called the **δ-weakening** of φ which is formally defined as follows:

$$\varphi^{-\delta} = Q_1^{[l_1,u_1]}x_1 ... Q_n^{[l_n,u_n]}x_n. \bigwedge_i \left(\bigvee_j f_{i,j}(\vec{x}) > -\delta \vee \bigvee_j f_{i,k}(\vec{x}) \geq -\delta \right)$$

- It is shown that this problem is **decidable** for signatures with computable functions [LICS12]

- The **complexity** for existential problems is **NP** (with P-time computable functions) or **PSPACE** (with Lipschitz ODEs) [LICS12]

Thesis Statement



"This thesis aims to **show the steps** that are taken **towards filling in this gap** with **convincing** and **practical examples** showing the **broad applicability** of these procedures."

Thesis Statement

Theoretical Possibility
(Decidability & Complexity)

⟷

Scalable & Practical
Delta-decision Procedures
(Algorithms & Implementations)

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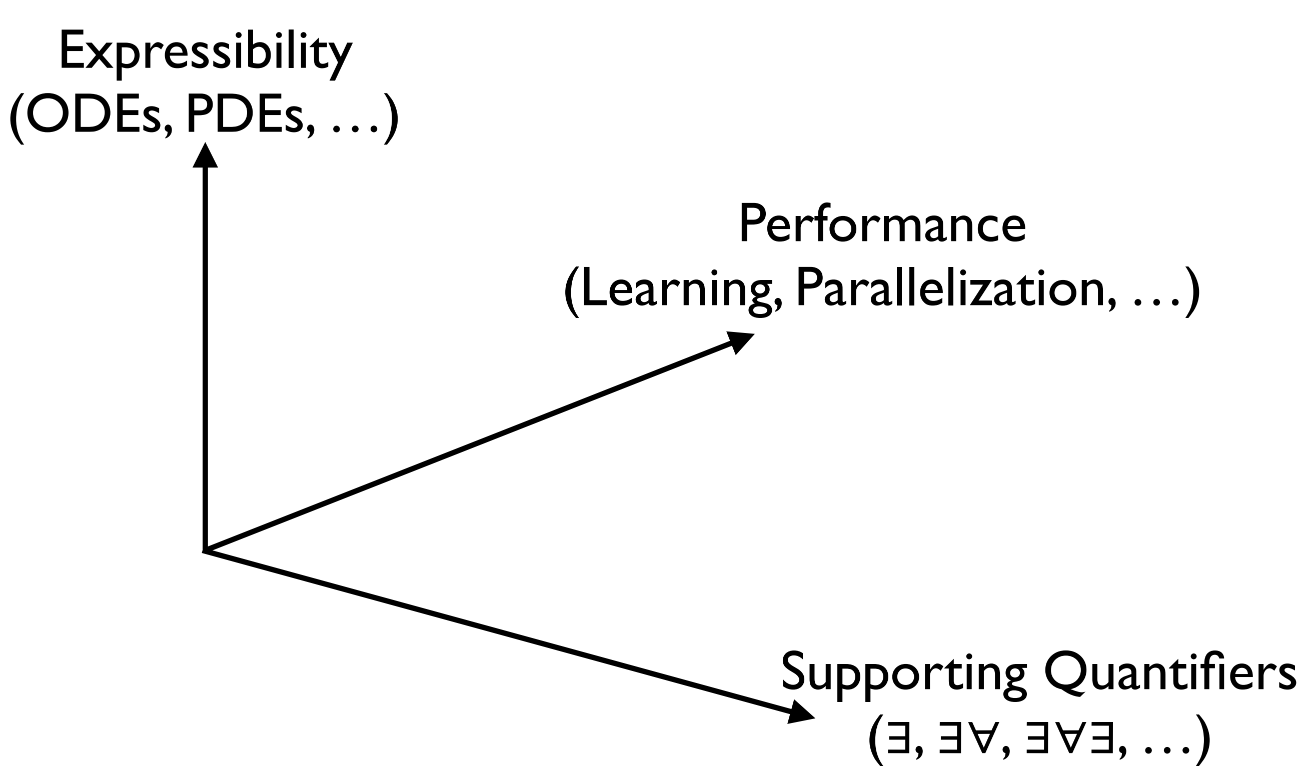

Thesis Statement

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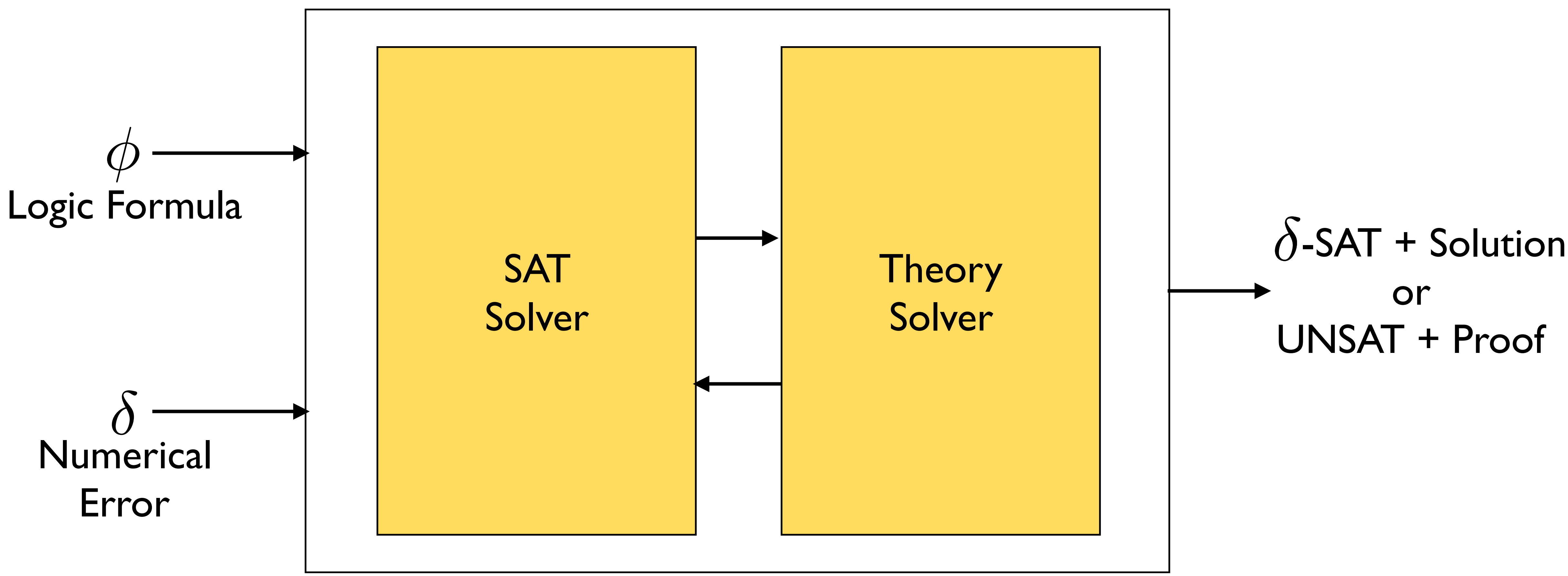
Research Questions:

- How to handle ODEs?
- How to integrate learning and non-chronological backtracking in solving?
- How to handle exist-forall problems and use the technique for optimization problems?

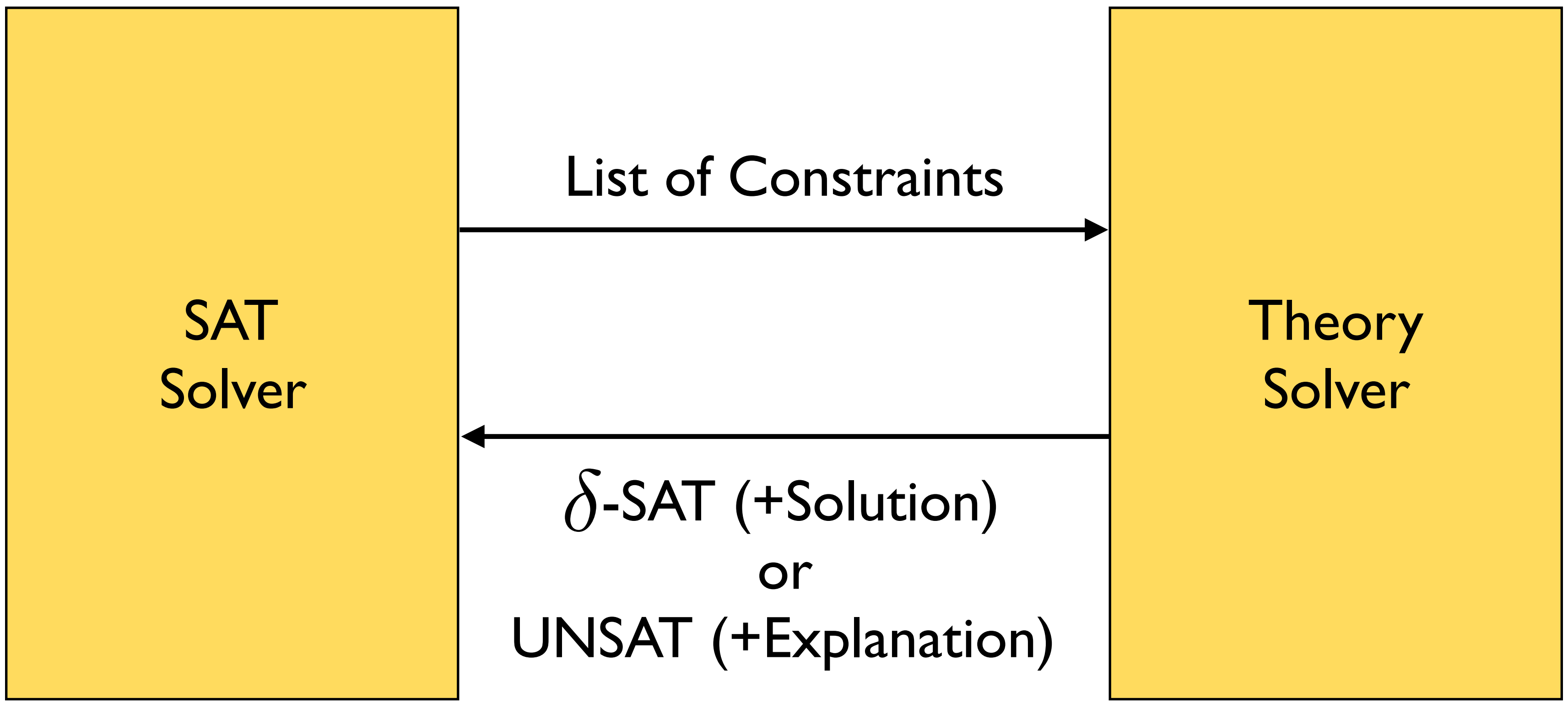
Chapter 2
Background

Design of Solver: Big Picture



- **SAT solver** finds a satisfying **Boolean** assignment
- **Theory solver** checks whether the assignment is feasible under the first order theory of **Real**

Design of Solver: Big Picture

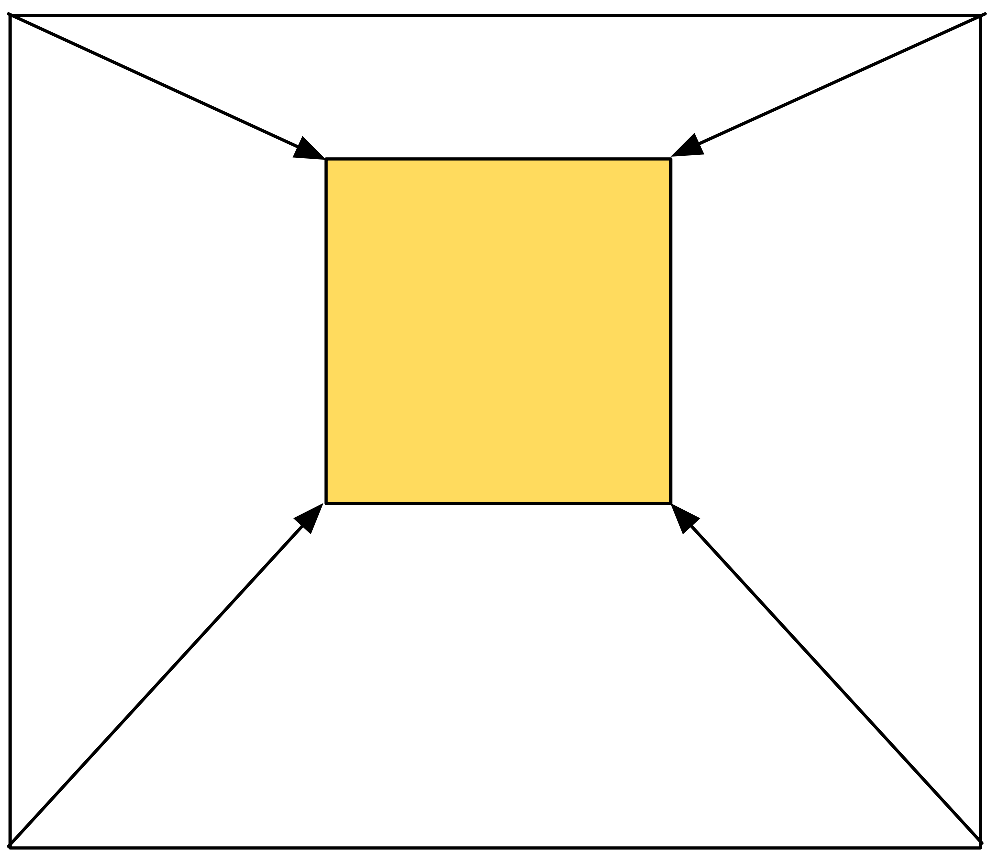


Boolean Search
Non-chronological Backtracking
Learning
…

(Discrete Domain)

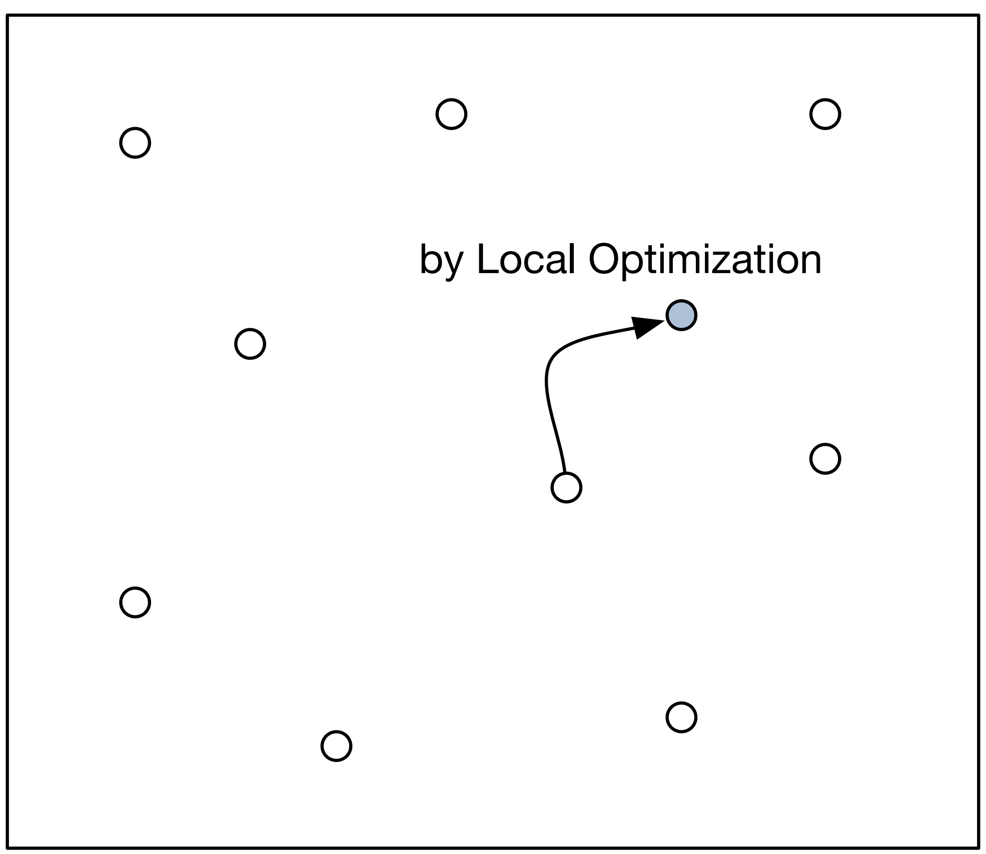
Constraints Solving
Validated Numerics
Optimization
Simulation/Sampling
…
(Continuous Domain)

Top-down/Bottom Approaches in Theory Solver



Top-Down Approach

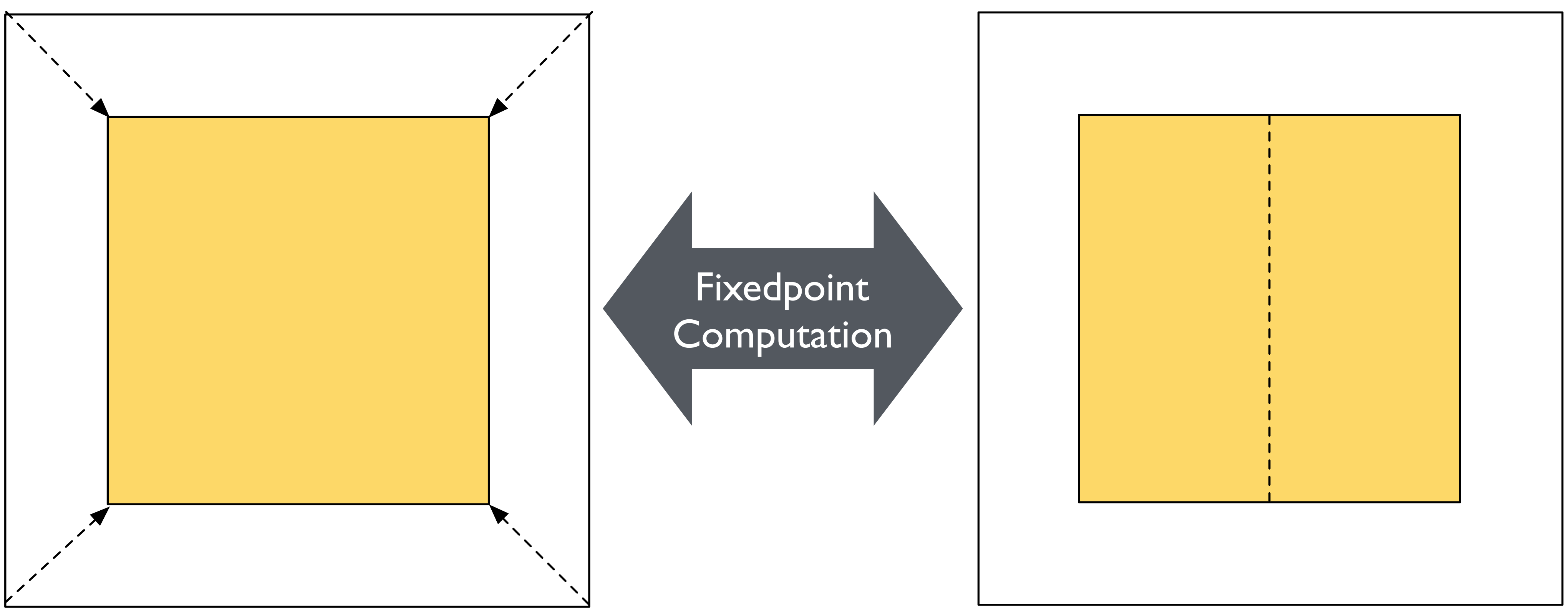
Maintain a set of possible solutions
Useful to show **UNSAT**
Validated Numerics
(i.e. Interval-based methods)



Bottom-Up Approach

Sample points and test them
Useful to show **SAT**
Use local-optimization to improve

An Algorithm in Theory Solver: ICP(Interval Constraint Propagation)



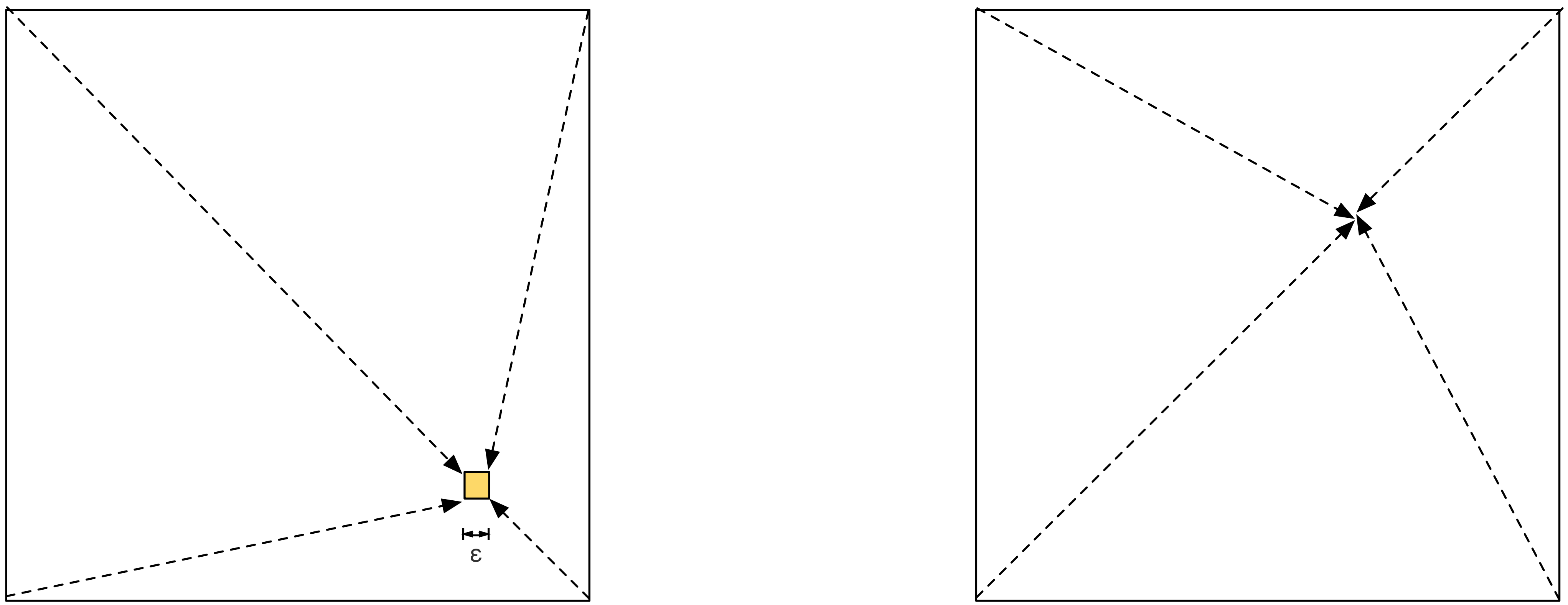
Pruning

Branch

Safely **reduce** a search space without removing solutions

Partition a search space into two sub-spaces

Two Termination Conditions of ICP



δ-sat

Unsat

ICP Algorithm

Algorithm 1: Theory Solving in DPLL$\langle$ICP$\rangle$

input : A conjunction of theory atoms, seen as constraints, $c_1(x_1, ..., x_n), ..., c_m(x_1, ..., x_n)$, the initial interval bounds on all variables $B^0 = I_1^0 \times \cdots \times I_n^0$, box stack $S = \emptyset$, and precision $\delta \in \mathbb{Q}^+$.

output: δ-sat, or unsat with learned conflict clauses.

S.push(B_0);
while $S \neq \emptyset$ **do**
    $B \leftarrow S$.pop() ;
    while $\exists 1 \leq i \leq m, B \neq \text{Prune}(B, c_i)$ **do**
        `//Pruning without branching, used as the assert() function.`
        $B \leftarrow \text{Prune}(B, c_i)$;
    end
    `//The` ε `below is computed from` δ `and the Lipschitz constants of functions beforehand.`
    if $B \neq \emptyset$ **then**
        if $\exists 1 \leq i \leq n, |I_i| \geq \varepsilon$ **then**
            $\{B_1, B_2\} \leftarrow \text{Branch}(B, i)$; `//Splitting on the intervals`
            S.push($\{B_1, B_2\}$);
        else
            return δ-sat; `//Complete check() is successful.`
        end
    end
end
return unsat;

Pruning

Branching

Chapter 3
Solving Delta-decision Problems with ODEs
[Completed Work]

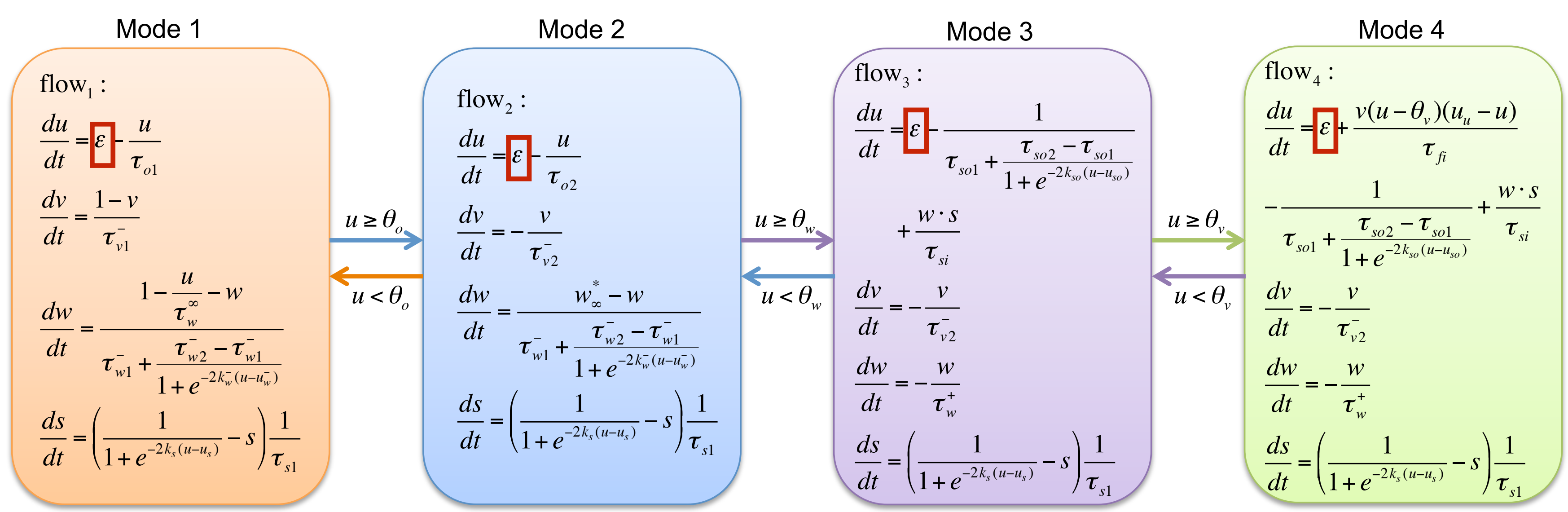
Solving Delta-decision Problems with ODEs

Motivation

- ODEs are widely used in the design and verification of Hybrid Systems (i.e. in Biomedical, Robotics).

- Most of them include highly-nonlinear dynamics.

Cardiac Cell Action Potential Model



Bing Liu, Soonho Kong, Sicun Gao, Paolo Zuliani, and Edmund Clarke,
"Parameter Synthesis for Cardiac Cell Hybrid Models Using Delta-Decisions.", CMSB 2014

Solving Delta-decision Problems with ODEs

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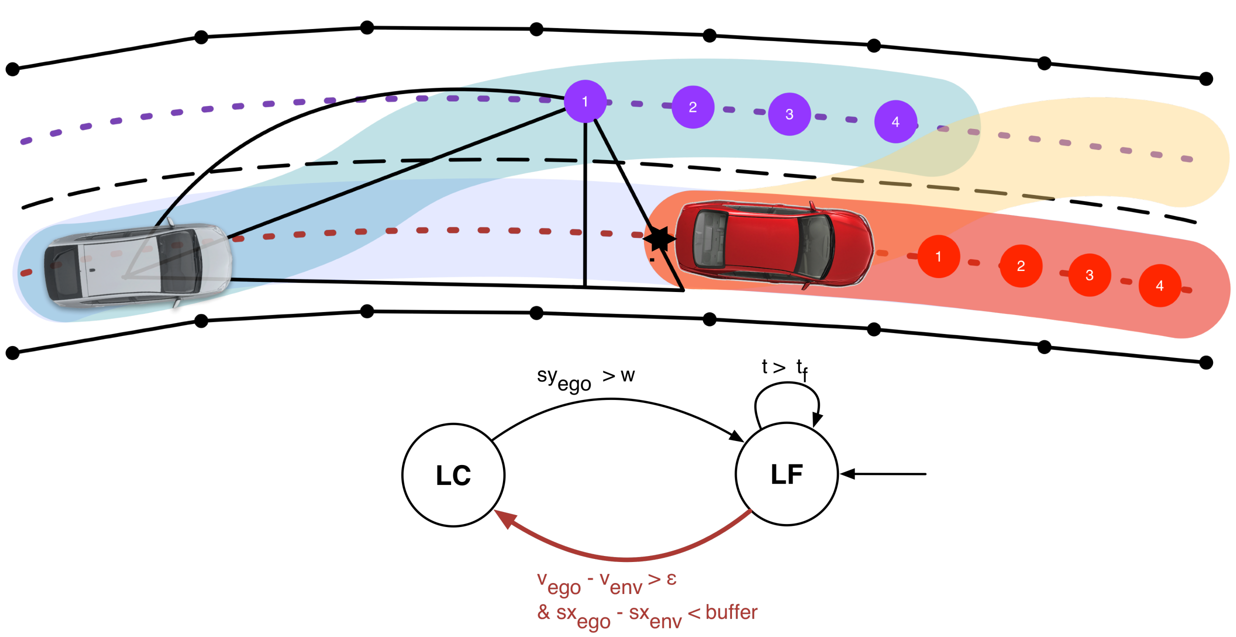
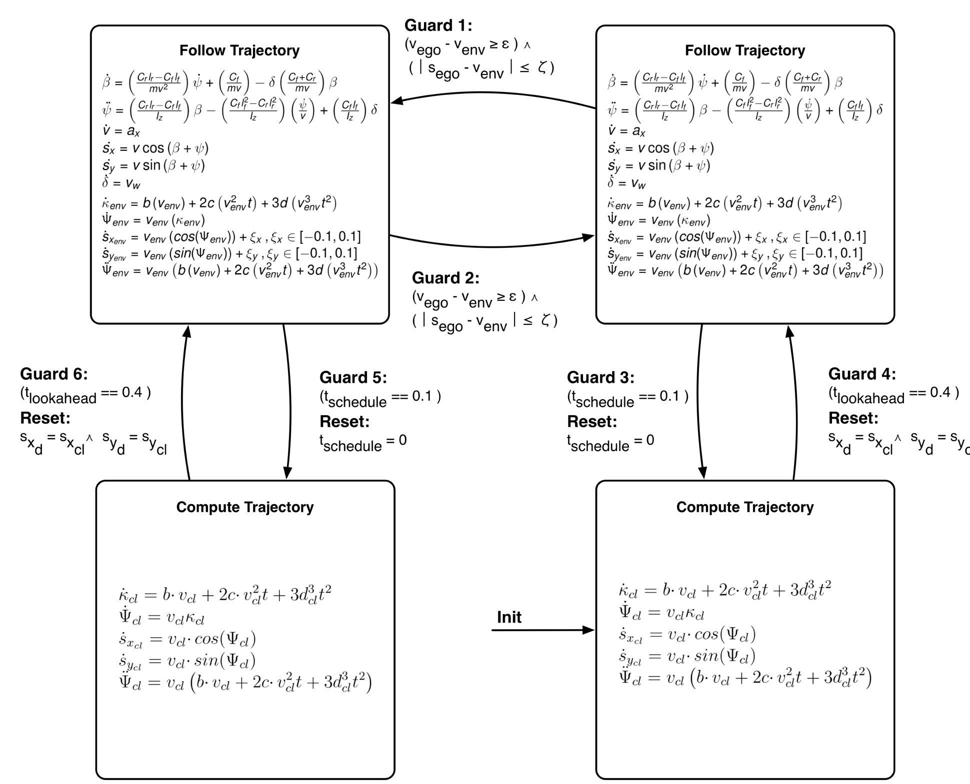
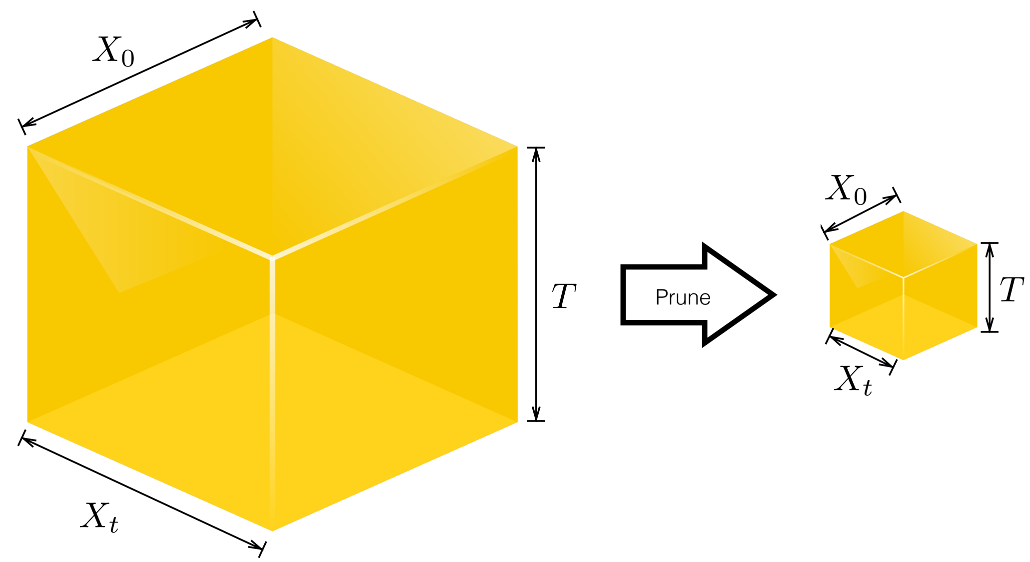




Matthew O'Kelly, Houssam Abbas, Sicun Gao, Shin'ichi Shiraishi, Shinpei Kato, and Rahul Mangharam ,
"APEX: A Tool for Autonomous Vehicle Plan Verification and Execution", In Society of Automotive Engineers (SAE) World Congress and Exhibition 2016

Solving Delta-decision Problems with ODEs

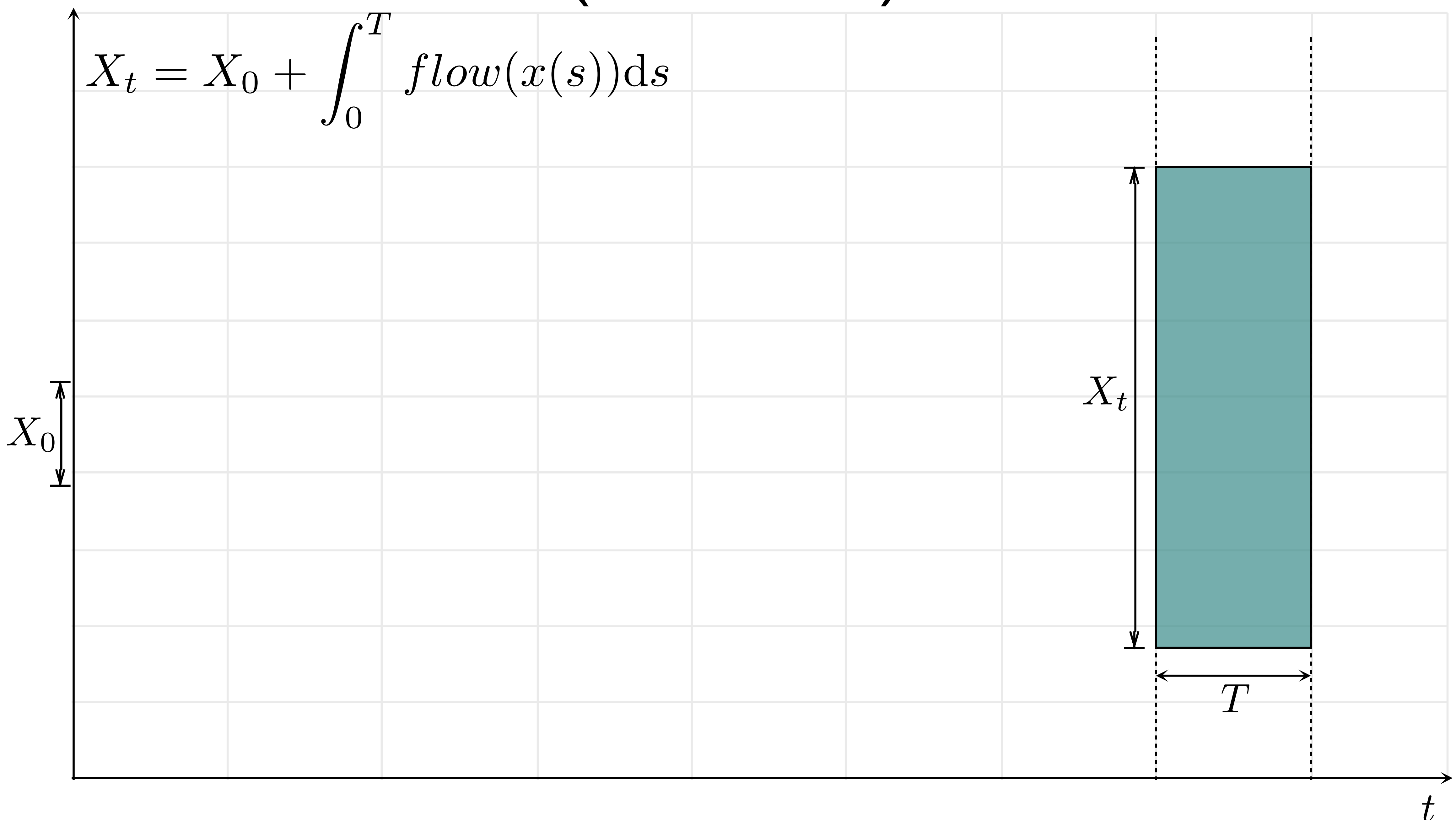
Approach



$$X_t = X_0 + \int_0^T flow(x(s))\mathrm{d}s$$

1. Design pruning operators from an ODE constraint.
2. Use rigorous numerical ODE solvers to propagate interval assignments on initial/final/time variables.

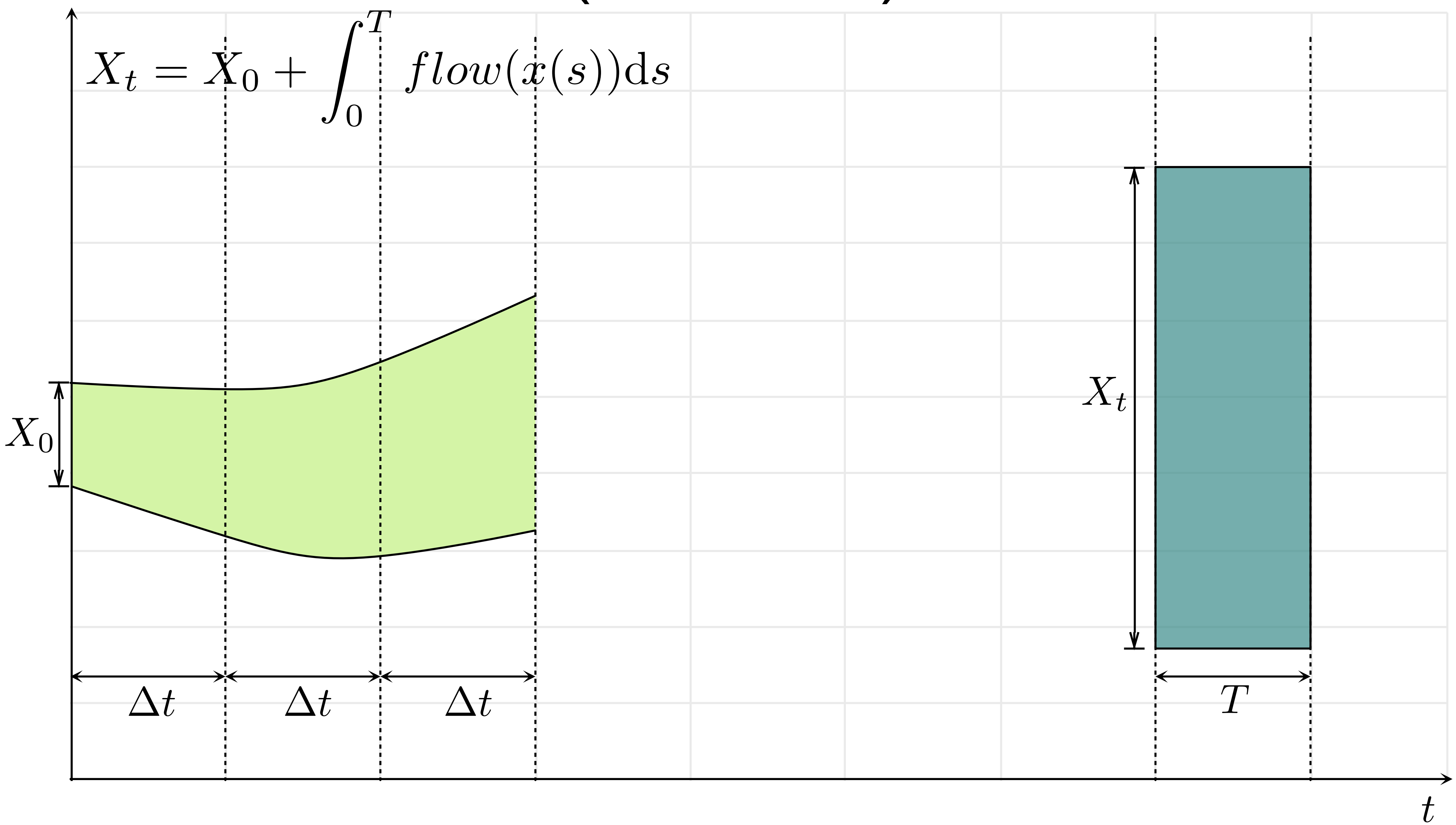
Pruning using ODEs (Forward)



pruning on Xt

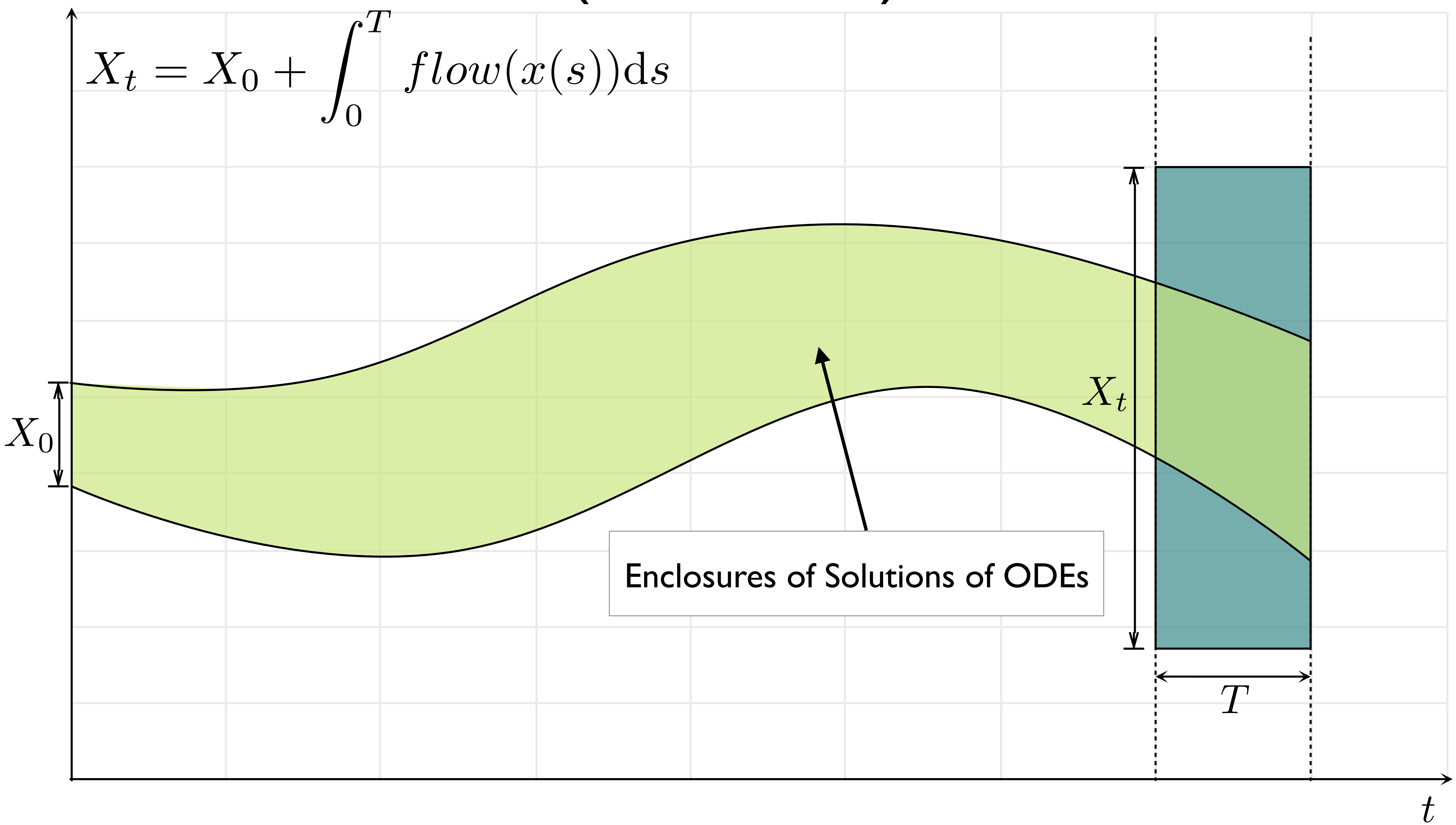
How can we prune X_t?

Pruning using ODEs (Forward)



(numerically) Compute the enclosures of the solutions of ODE

Pruning using ODEs (Forward)



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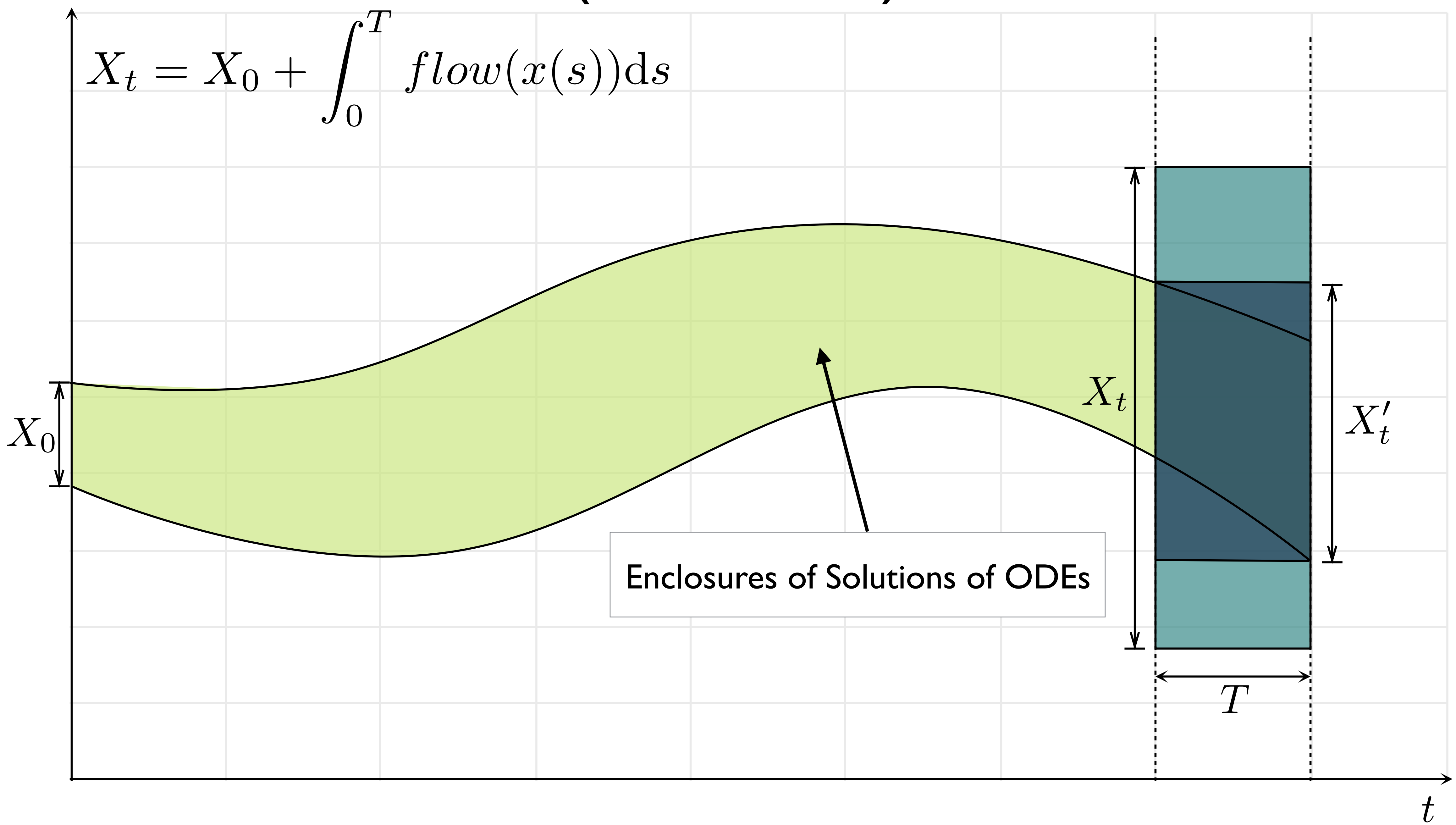
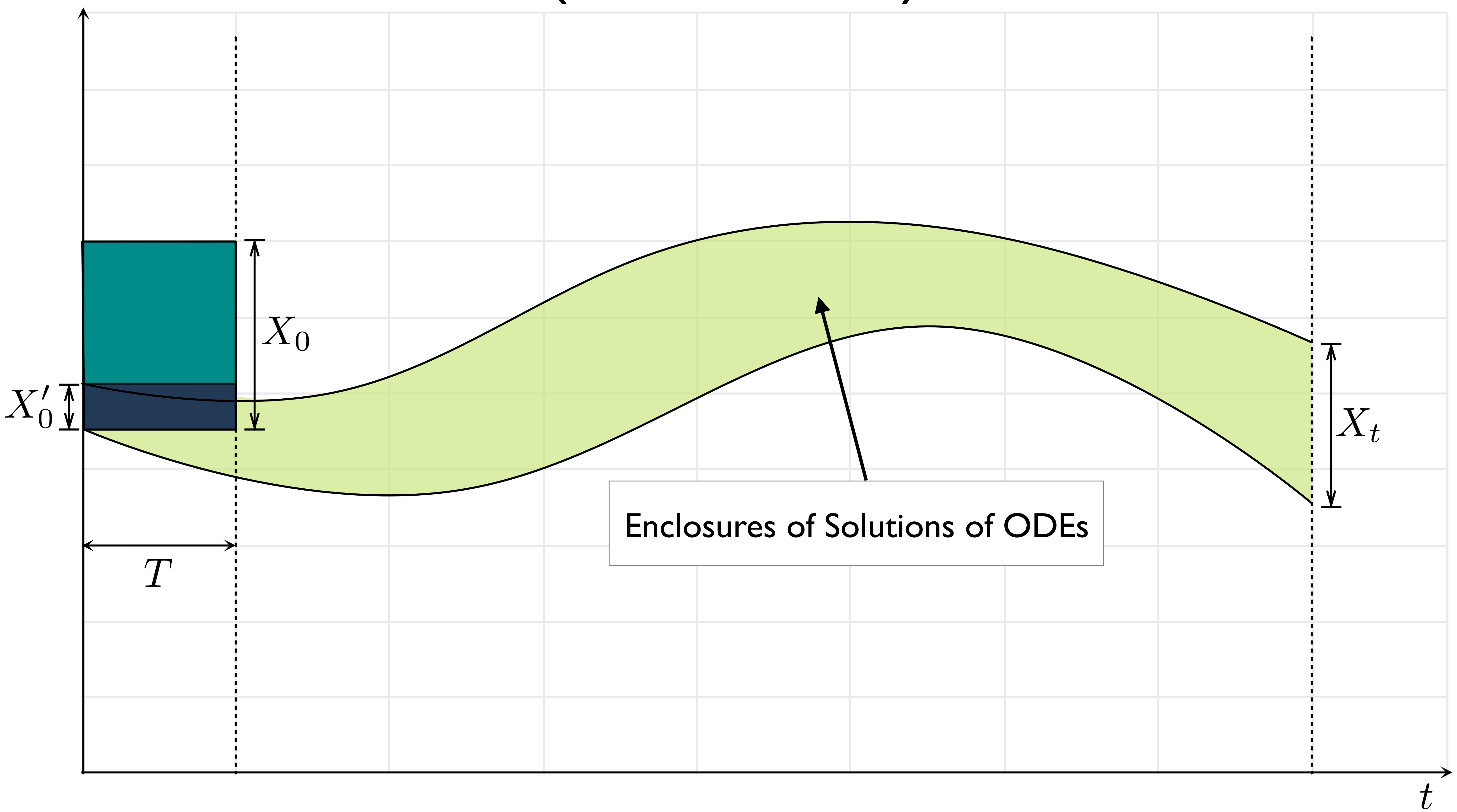


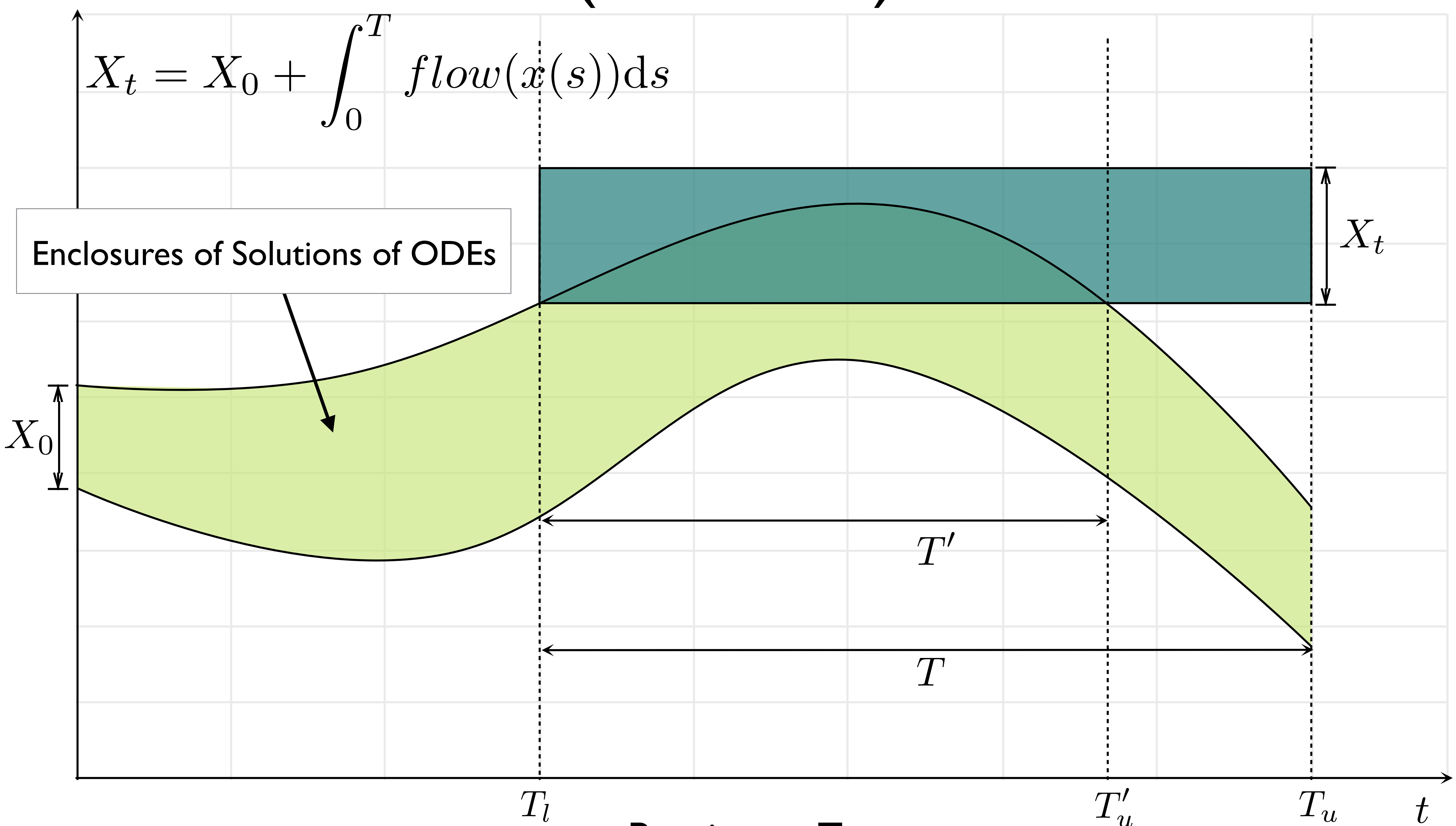
Take the intersection between the Enclosure and Xt

Pruning using ODEs (Backward)



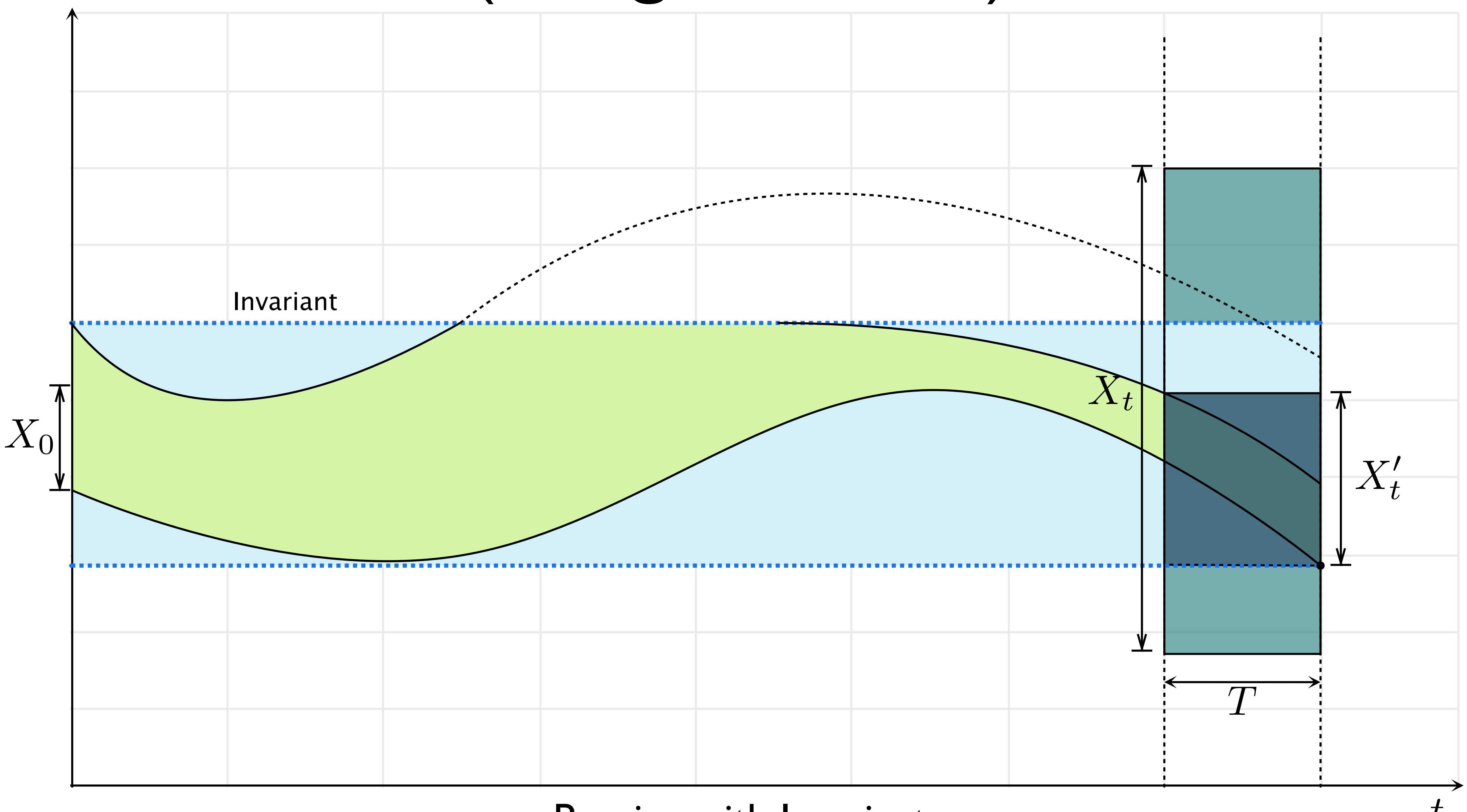
Pruning on X0

Pruning using ODEs (on Time)



Pruning on T

Pruning using ODEs
(using Invariant)



Pruning with Invariant

Solving Delta-decision Problems with ODEs
Result

* Implemented in **dReal**
* Can handle a formula with 250+ ODEs and 600+ Vars
* Published a **paper** in FMCAD'13
* There are **applications** and **tools** based on this technique

Solving Delta-decision Problems with ODEs
Result

Applications:

* Autonomous Driving (Penn) [SAE'16]
* Planning (CMU,SIFT) [AAAI'15]
* Atrial Fibrillation (Stony Brook, TU, CMU) [HSCC'15,CMSB'14]
* Diabetes (Penn) [ADHS'15]
* Prostate Cancer (Pitt, CMU) [HSCC'15]

Tools based on dReal:

* APEX: A Tool for Autonomous Vehicle Plan Verification and Execution (Toyota/UPenn)
* BioPSy: Parameter set synthesis on biological models (Univ. of Newcastle)
* dReach: Reachability analysis tool for hybrid system (CMU)
* ProbReach: Probabilistic reachability analysis of hybrid systems (Univ. of Newcastle)
* SReach: Bounded model checker for stochastic hybrid systems (CMU)

Chapter 4
SAT-driven Branch-and-Prune
[Work in Progress]

SAT-driven Branch-and-Prune
Motivation

	SAT Solving (DPLL/CDCL)	ICP
Search	Split Rule (i.e. Making a decision)	Branching
Inference	Unit Resolution (Boolean Constraint Propagation)	Pruning

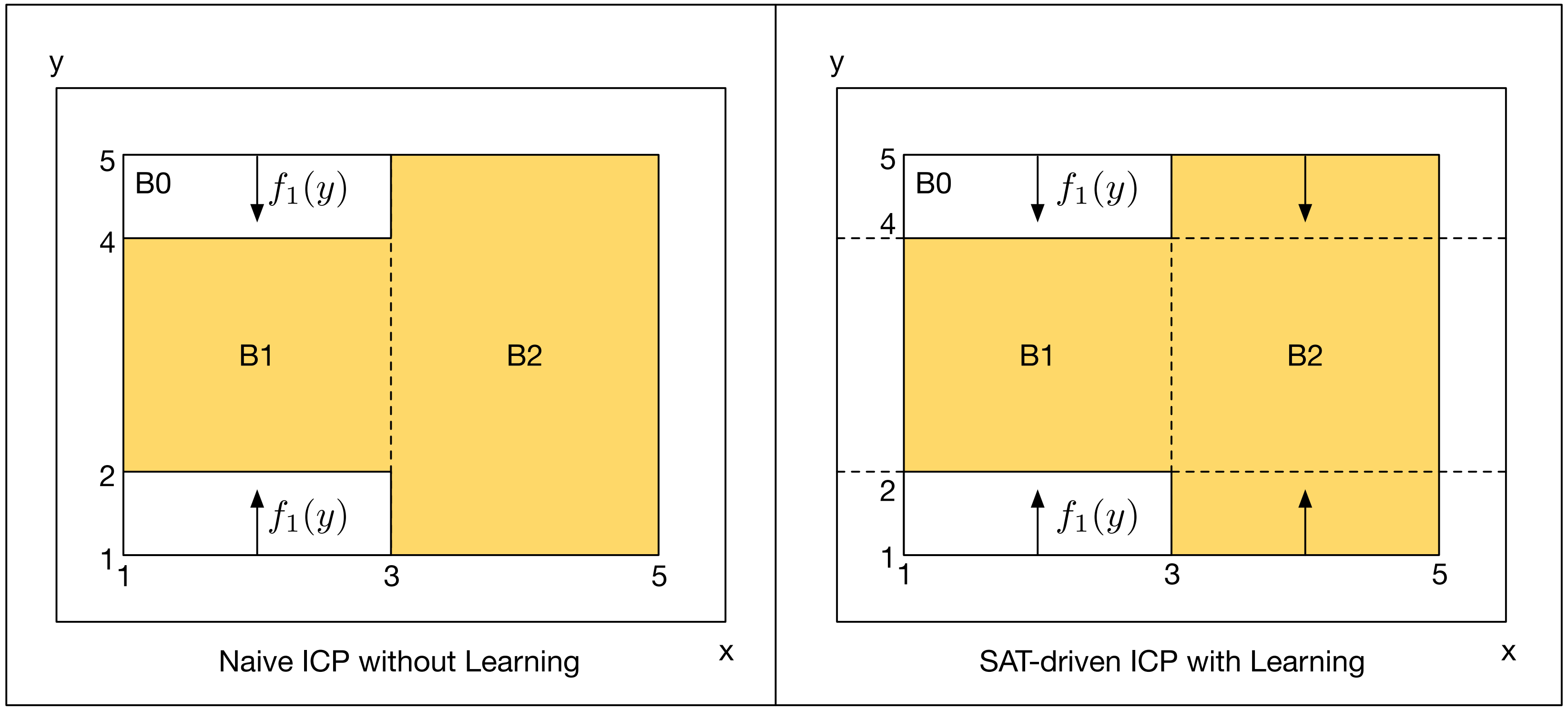
SAT-driven Branch-and-Prune

Motivation

	SAT Solving (DPLL/CDCL)	ICP
Search	Split Rule (i.e. Making a decision) Conflict Clause Learning + Non-chronological Backtracking	Branching Depth-first Search without Learning Chronological Backtracking with Stack
Inference	Unit Resolution (Boolean Constraint Propagation)	Pruning

SAT-driven Branch-and-Prune

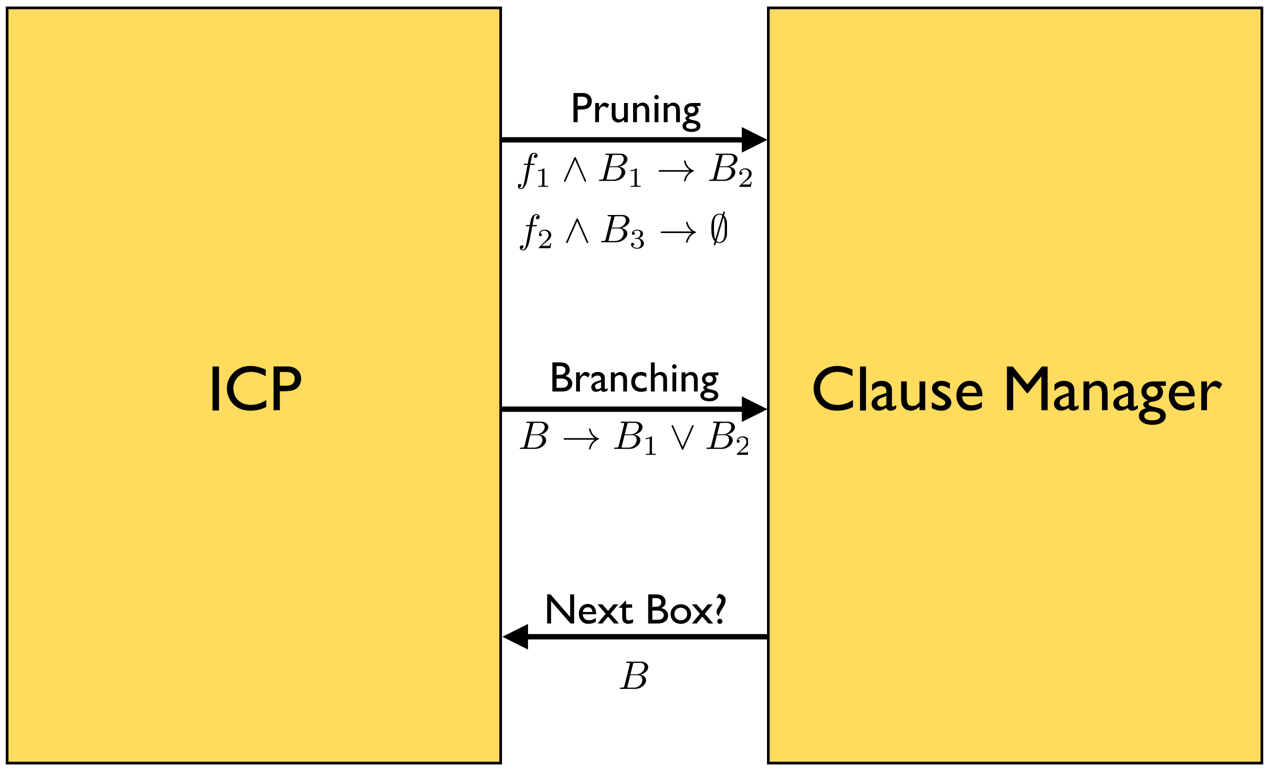
Motivation



Pruning: $f_1(y) \wedge \{x : [1,3], y : [1,5]\} \rightarrow \{x : [1,3], y : [2,4]\}$

Learned Clause: $f_1(y) \wedge \{y : [1,5]\} \rightarrow \{y : [2,4]\}$
(after generalization)

SAT-driven Branch-and-Prune Approach

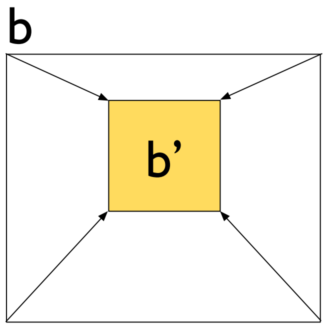


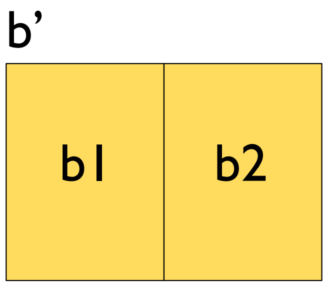
ICP → Clause Manager : Report **pruning/branching** steps

Clause Manager → ICP : Provide the **next box** to visit

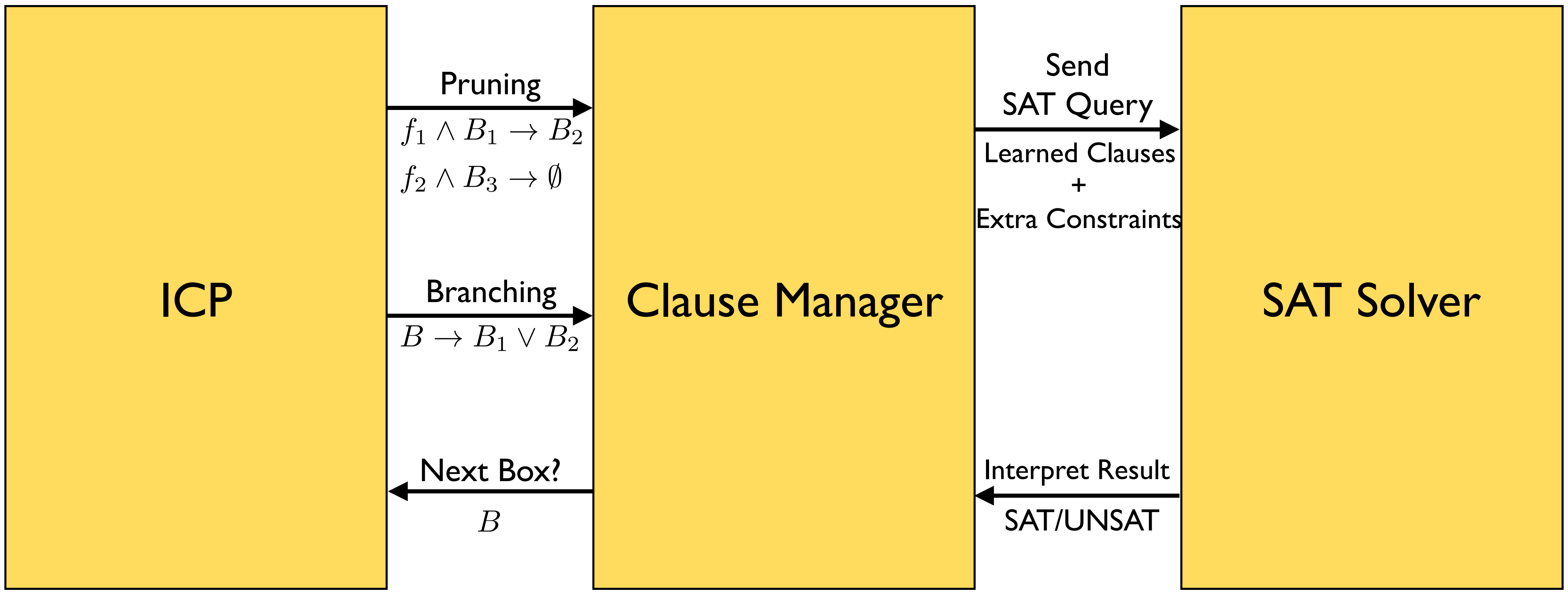
SAT-driven Branch-and-Prune Approach

```
SAT_ICP(Constraint f, Box b) {
    CM.init(b);  // set initial search space
    while (b = CM.next_box()) {
        // Pruning
        do {
            b′ = f.prune(b);
            CM.learn(f, b → b′);
        } while (b′ ≠ ∅ ∧ b′ ≠ b);
        if (b′ = ∅) {
            break;  // try to get a new box
        }
        if (|b′| ≤ ε) {
            return δ-SAT(b′);
        }
        // Branching
        (b1, b2) = branch(b′);
        CM.learn(b′ → b1 ∨ b2);
        b = b1;  // search b1 first
    }
    return UNSAT;  // no box to search
}
```



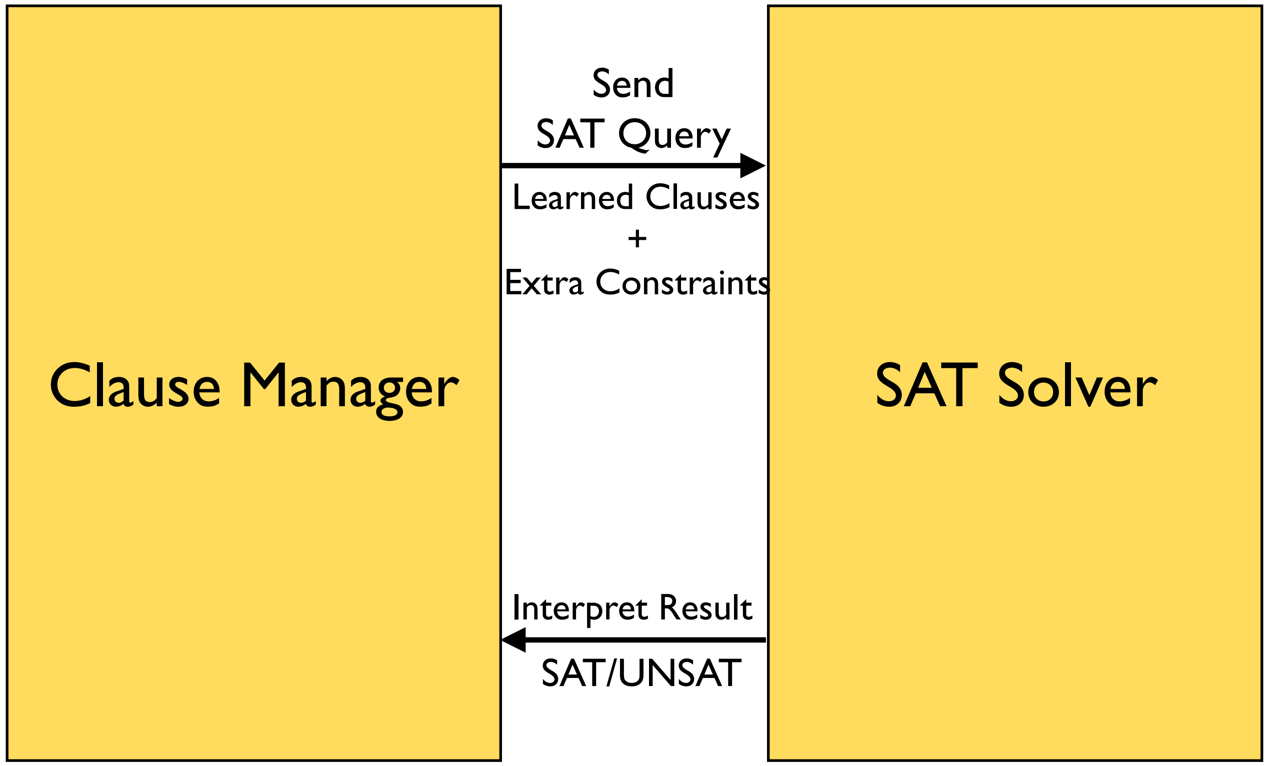


SAT-driven Branch-and-Prune Approach



SAT-driven Branch-and-Prune

Approach



Boolean Encoding

1. To each predicate (x ≥ c) (resp. x ≤ c), **associate a Boolean variable** $b_{(x \geq c)}$ (resp. $b_{(x \leq c)}$)

$$\{x : [1,3], y : [1,5]\} \quad = \quad (1 \leq x) \wedge (x \leq 3) \wedge (1 \leq y) \wedge (y \leq 5)$$

$$\downarrow$$

$$b_{1 \leq x} \wedge b_{x \leq 3} \wedge b_{1 \leq y} \wedge b_{y \leq 5}$$

2. Introduce **Extra Constraints**

Ordering Constraints: $(x \leq 1) \to (x \leq 3)$ $\quad (x \geq 3) \to (x \geq 1)$

Disjointness Constraints: $(x \geq 3) \to \neg(x \leq 1)$

SAT-driven Branch-and-Prune Approach

Clause Manager

Simplification of Clauses

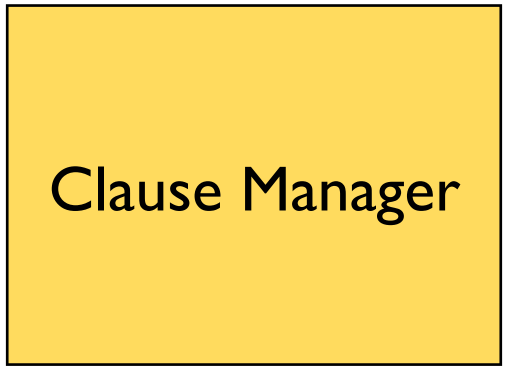
1. Using **resolution rule** to infer new clauses

$$\frac{b_1 \rightarrow b_2 \qquad \neg b_2}{\neg b_1}$$

2. Using **subsumption rule** to eliminate redundant clauses

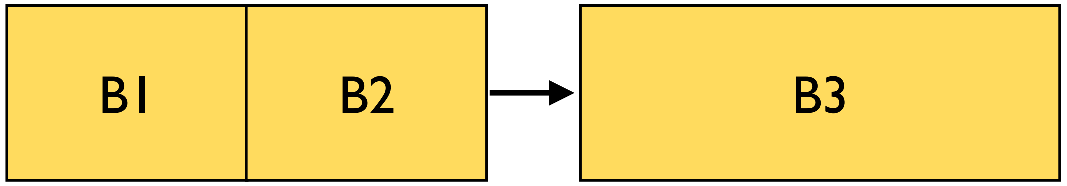
$$\{b_1 \rightarrow b_2, \neg b_2, \neg b_1\} \quad \Longrightarrow \quad \{\neg b_1\}$$

SAT-driven Branch-and-Prune Approach

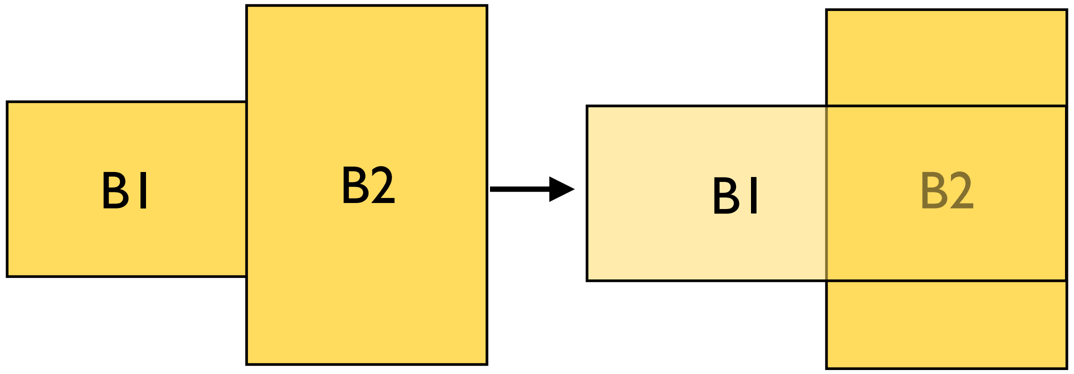


Simplification of Clauses

3. Replacing **two adjacent boxes** with a single box by **merging** them



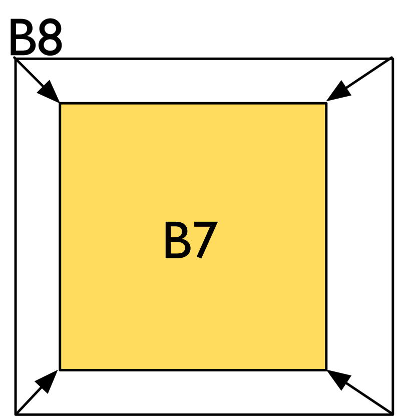
4. Relaxing/enlarging a box using its neighbors



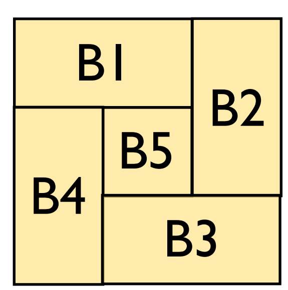
SAT-driven Branch-and-Prune Approach

Clause Manager

An example:



B8 → B7



SAT-driven Branch-and-Prune Approach

Clause Manager

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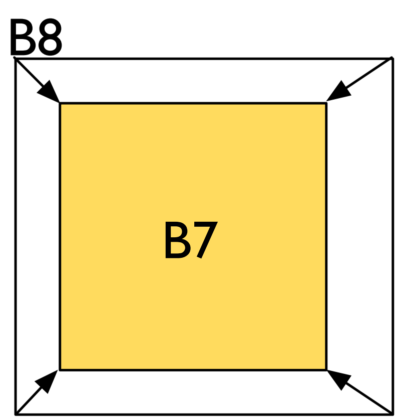

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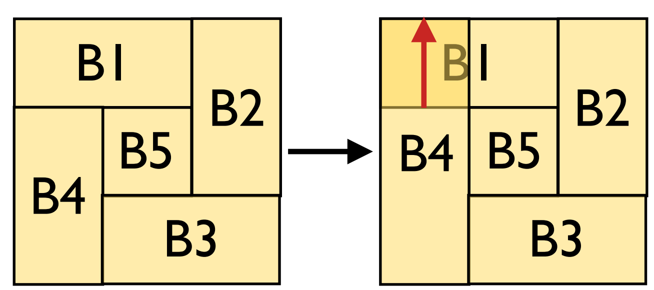
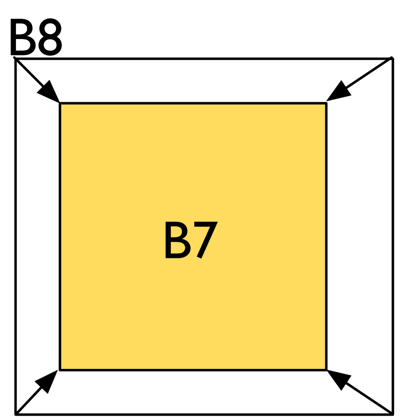


SAT-driven Branch-and-Prune

Approach

Clause Manager

An example:



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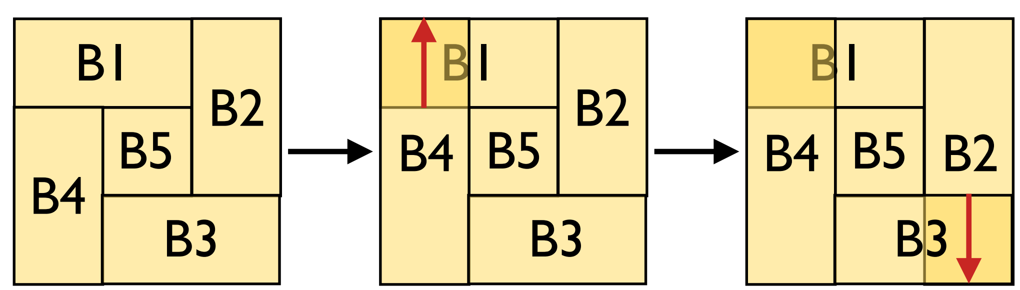
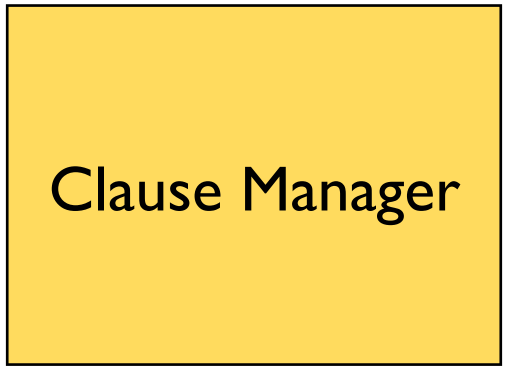


SAT-driven Branch-and-Prune Approach



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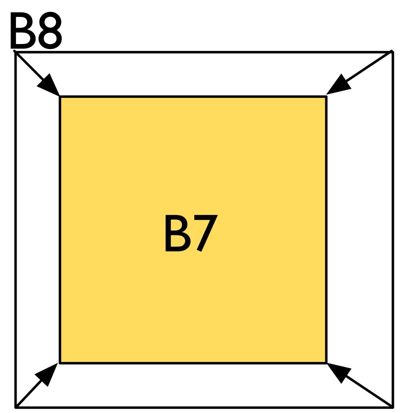

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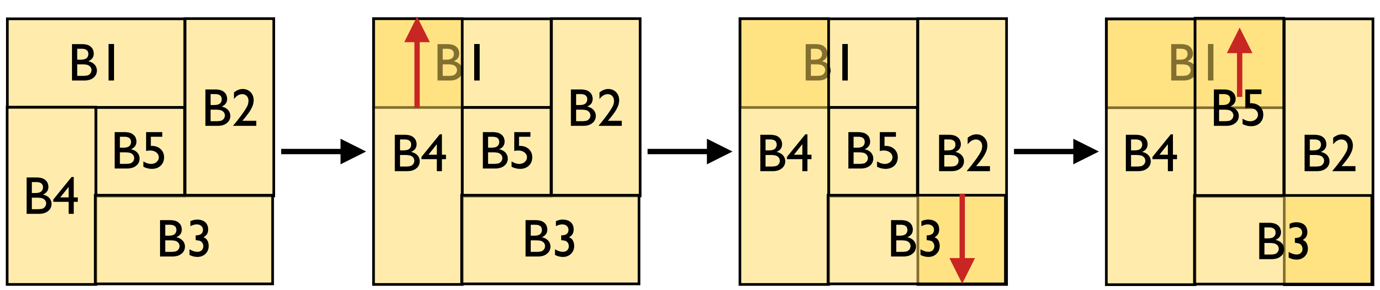
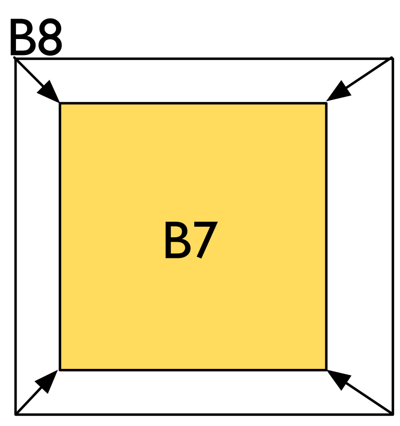


SAT-driven Branch-and-Prune Approach

Clause Manager

An example:



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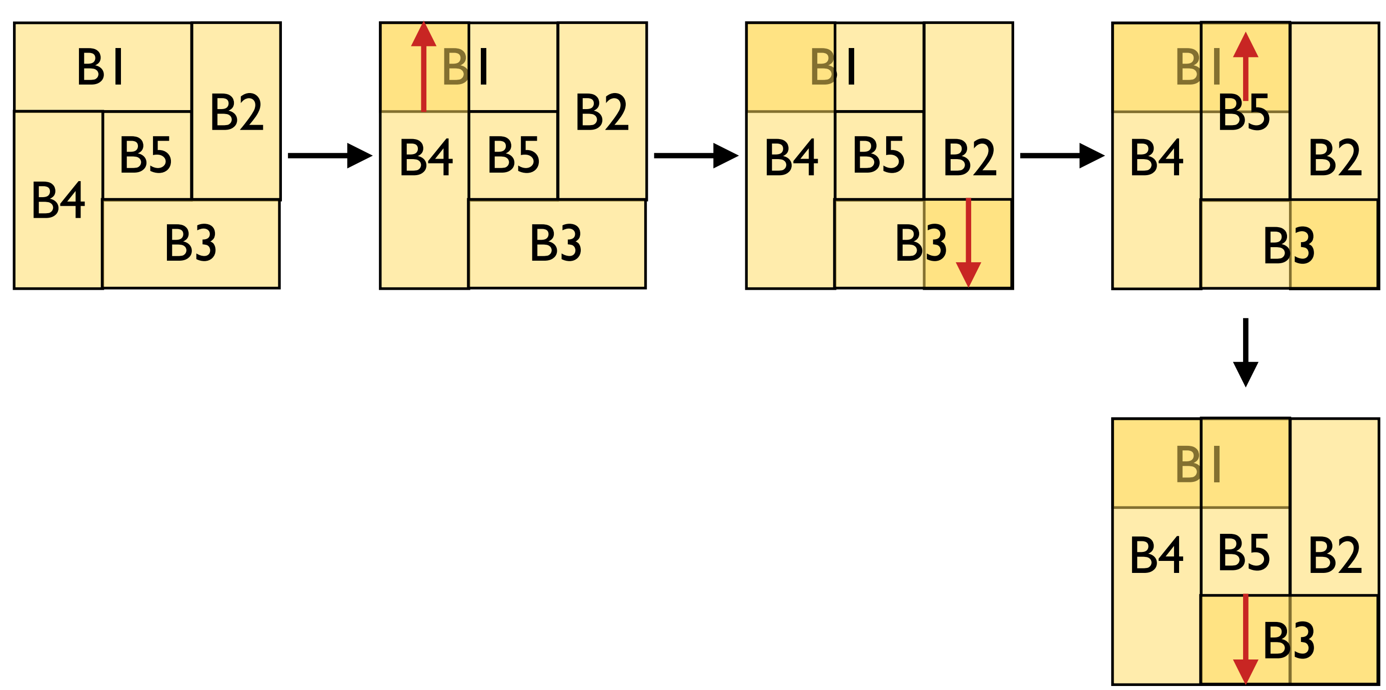

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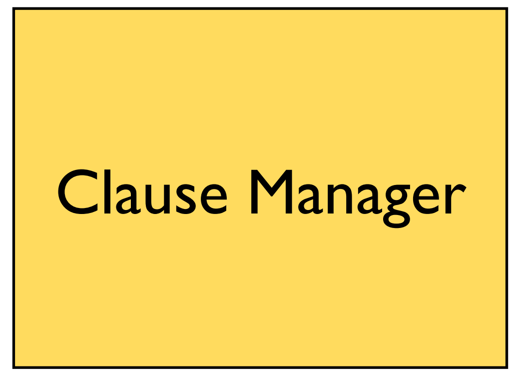

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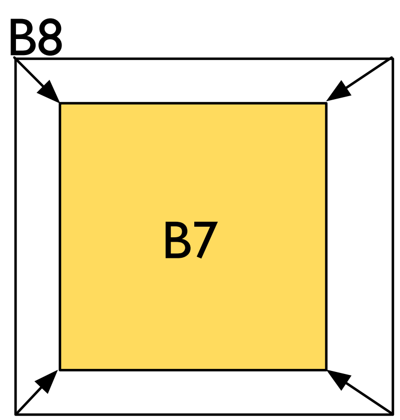

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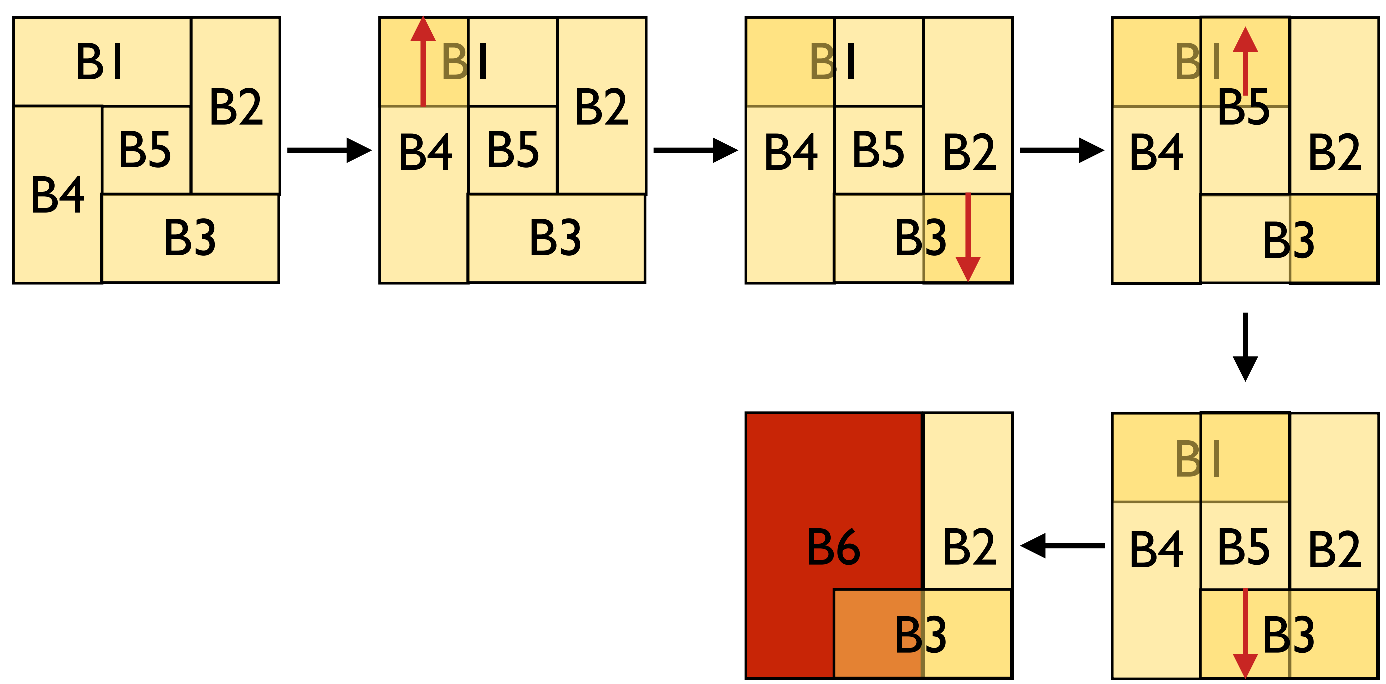

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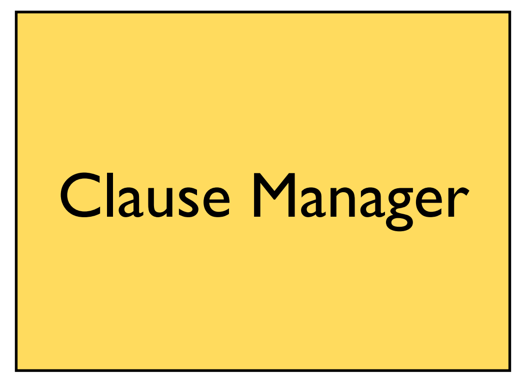

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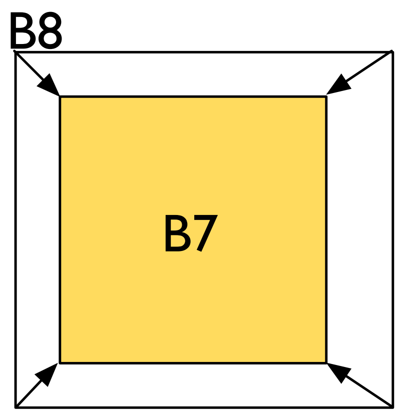
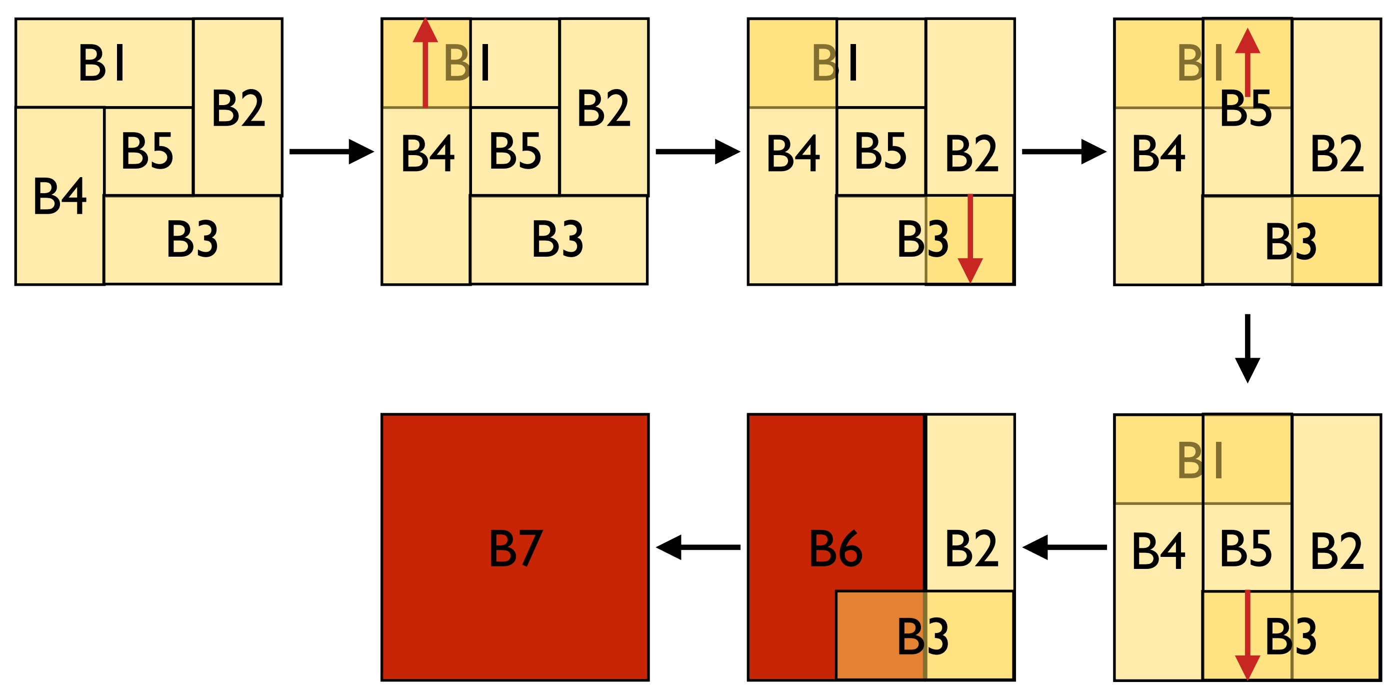


B8 → B7



SAT-driven Branch-and-Prune

Proposed Work

- Prove SAT+ICP algorithm **terminates**.
- Prove **correctness** of SAT+ICP algorithm.
  The outputs from naive ICP and SAT+ICP should be identical.
- Show that SAT+ICP algorithm **outperforms** naive ICP.
- Use **Boxes/LDD**[*] data structure to implement Clause Manager and check the performance gain.

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* Arie Gurfinkel, Sagar Chaki, "Boxes: A Symbolic Abstract Domain of Boxes". SAS 2010: 287-303

Chapter 5
Solving Exist-forall Formulas
[Work in Progress]

Solving Exist-forall Formulas

Motivation

$$\left\{\begin{array}{l}\text{Global}\\\text{Non-convex}\\\text{Non-linear}\\\text{Multi-objective}\\\text{Combinatorial (disjunctions)}\end{array}\right\}\ \text{Optimization}$$

Challenging Problems in Optimization

Solving Exist-forall Formulas

Approach

Encode optimization problems into first-order formula over Real with **one alternation of quantifiers**

$$\min f(\mathbf{x}) \text{ s.t. } \phi(\mathbf{x})$$

$$\Updownarrow \text{ is logically}$$

$$\boxed{\exists \mathbf{x}. \forall \mathbf{y}.}\ \phi(\mathbf{x}) \wedge \phi(\mathbf{y}) \rightarrow f(\mathbf{x}) \leq f(\mathbf{y})$$

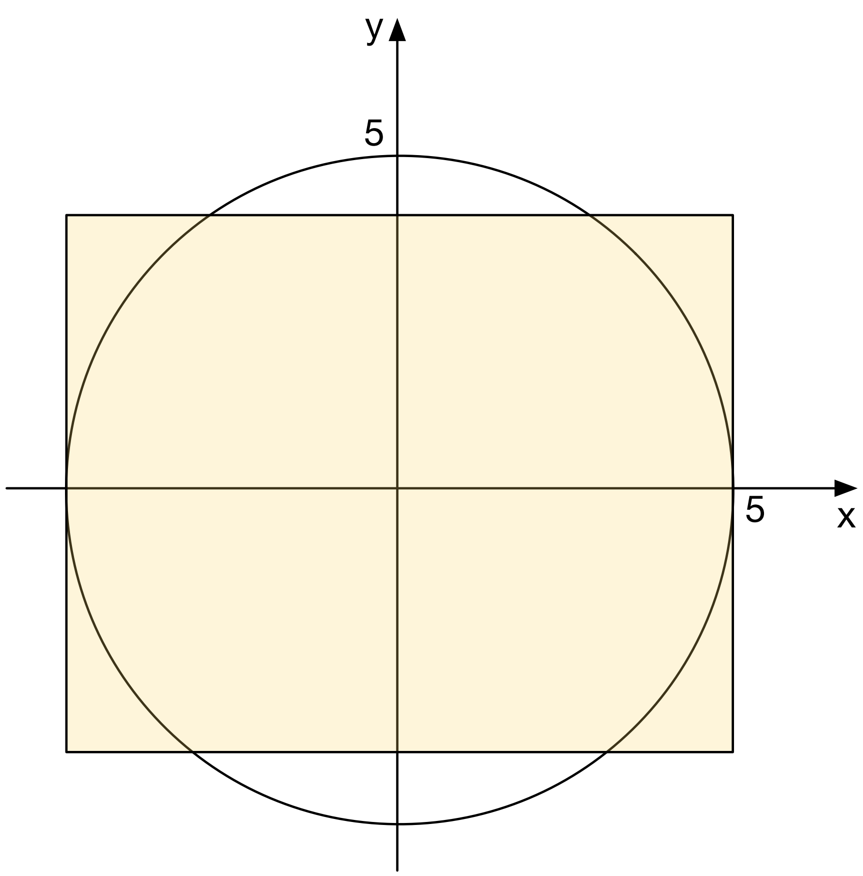
and **solve** exist-forall problems.

Solving Exist-forall Formulas

Approach

An example:

$$\exists^{[-5,5]} x. \forall^{[-4,4]} y.\ x^2 + y^2 <= 5^2$$



(A) Initial Search Space: x = [-5, 5]

Solving Exist-forall Formulas Approach

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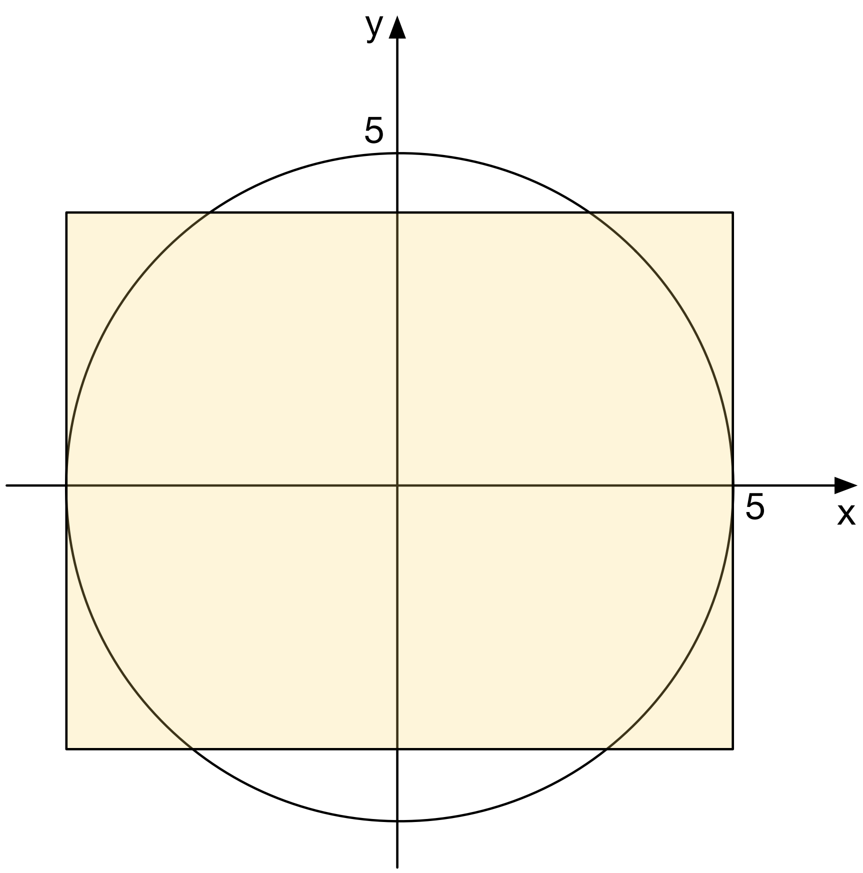

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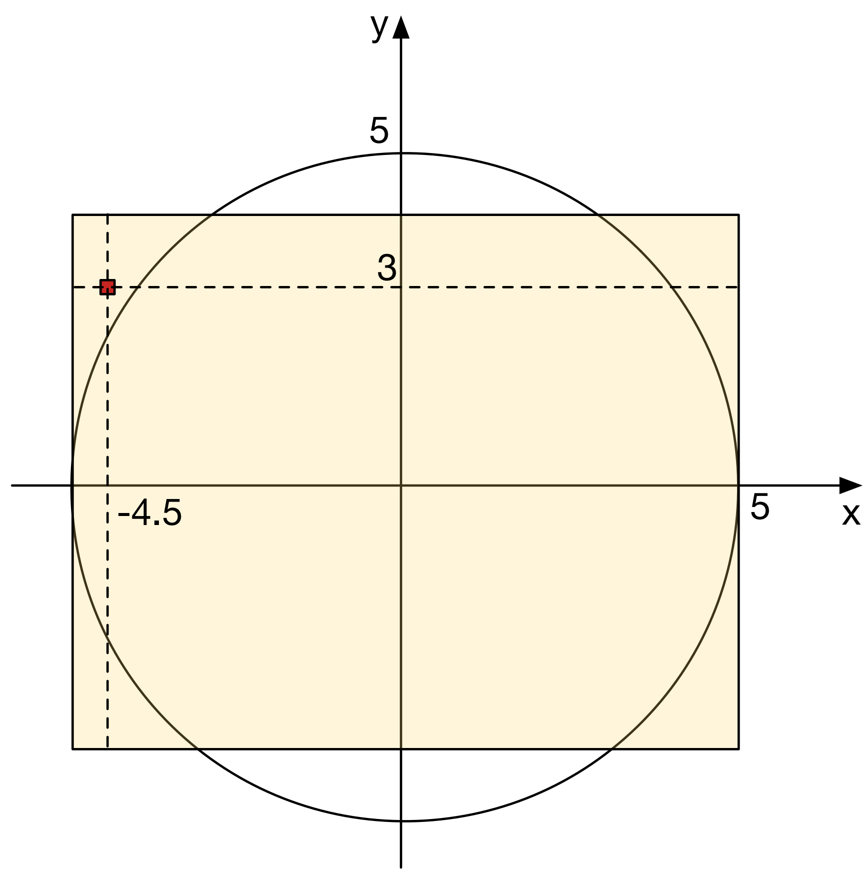

(B) Find a counterexample

$x = -4.5$

$y = 3$

Solving Exist-forall Formulas

Approach

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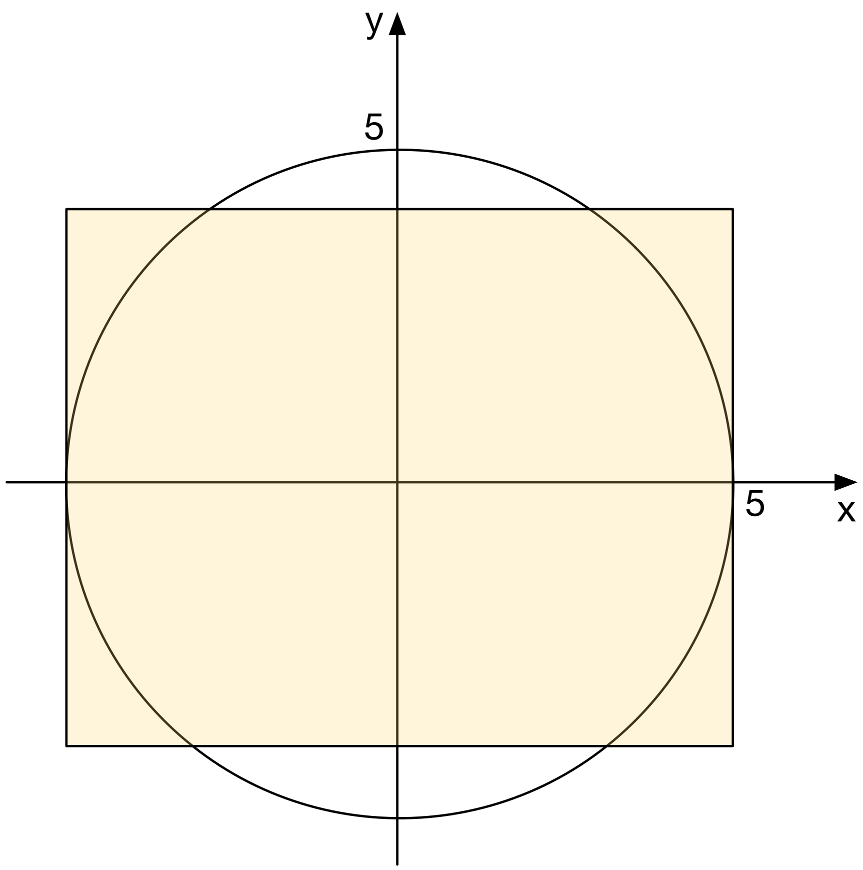

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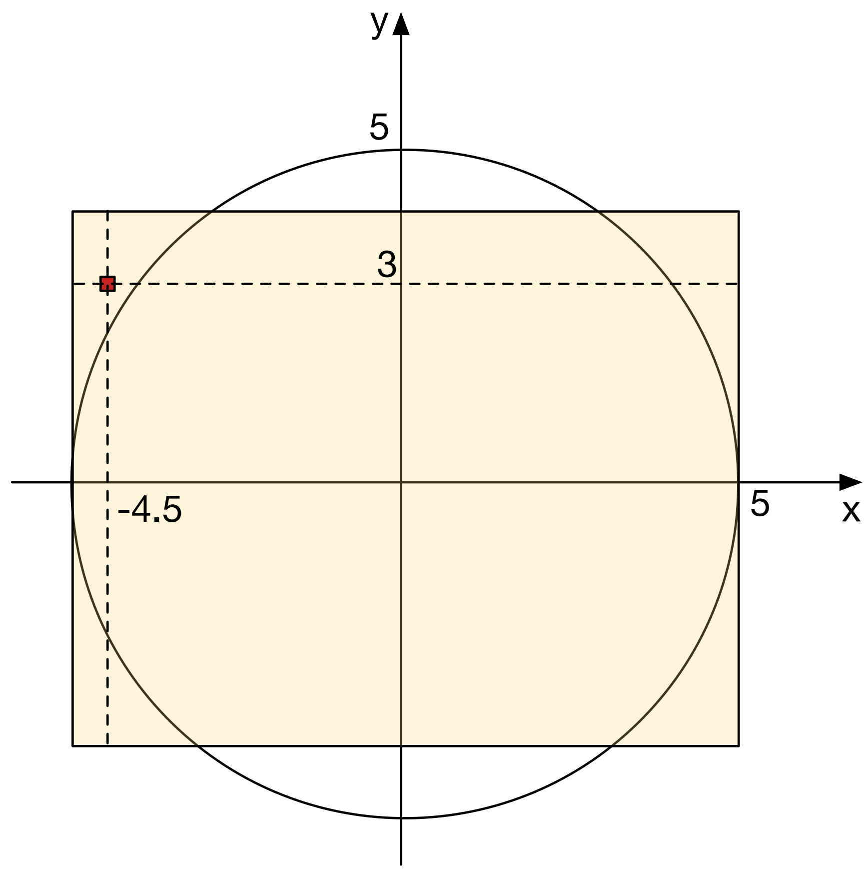

(B) Find a counterexample

$$x = -4.5$$
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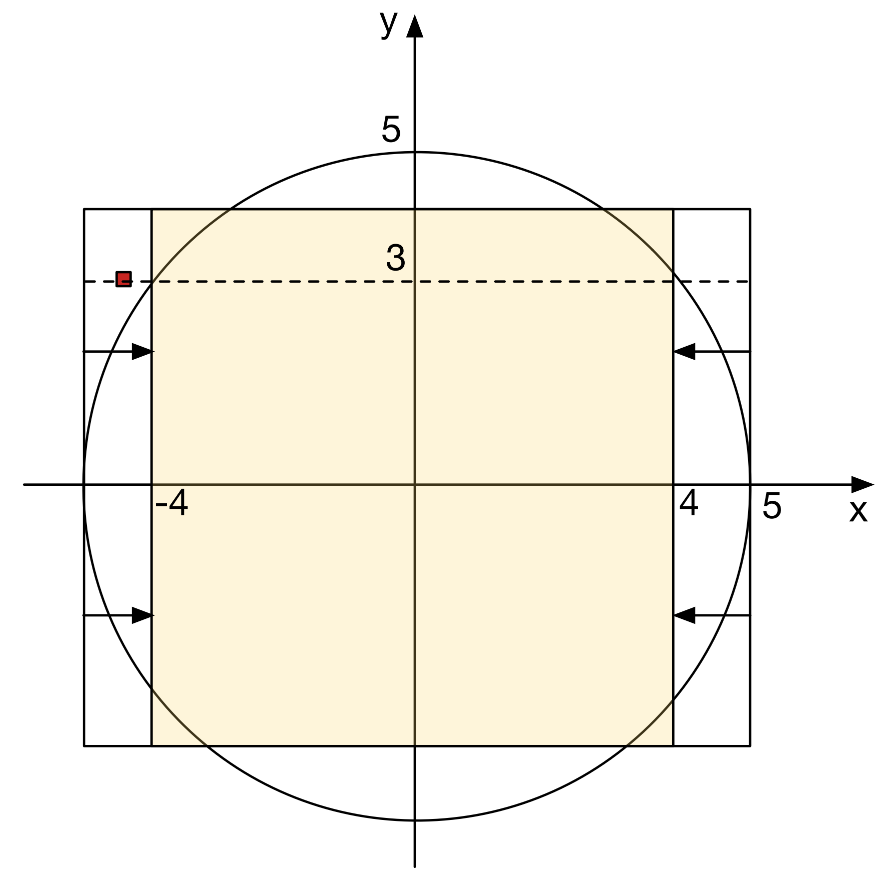


(C) Prune x using the counterexample

$$x^2 + \boxed{3^2} \leq 5^2$$
$$x^2 \leq 5^2 - 3^2 = 16 = 4^2$$
$$-4 \leq x \leq 4$$

Solving Exist-forall Formulas
Approach

Counterexample-guided Pruning Algorithm for exist-forall Problem

$$\exists x.\forall y.\ \varphi(x, y)$$

1. Counterexample generation:

$$b \leftarrow \mathrm{Solve}(y, \neg\varphi(x, y))$$

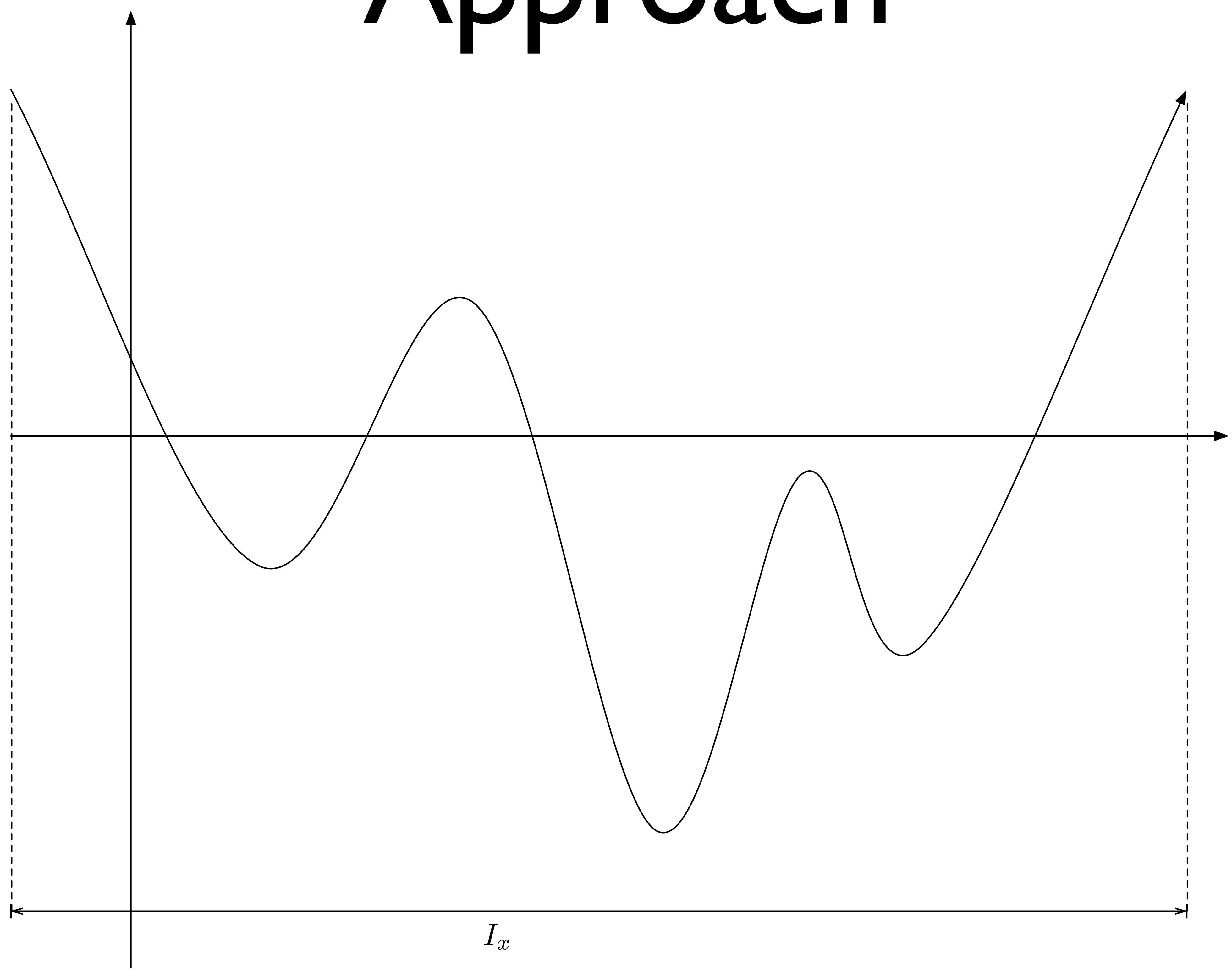
2. Pruning on x using the counterexample y = b:

$$B_x \leftarrow \mathrm{Prune}(B_x, \neg\varphi(x, b))$$

Repeat until it fails to find a counterexample in step 1 or reaches a fixedpoint.

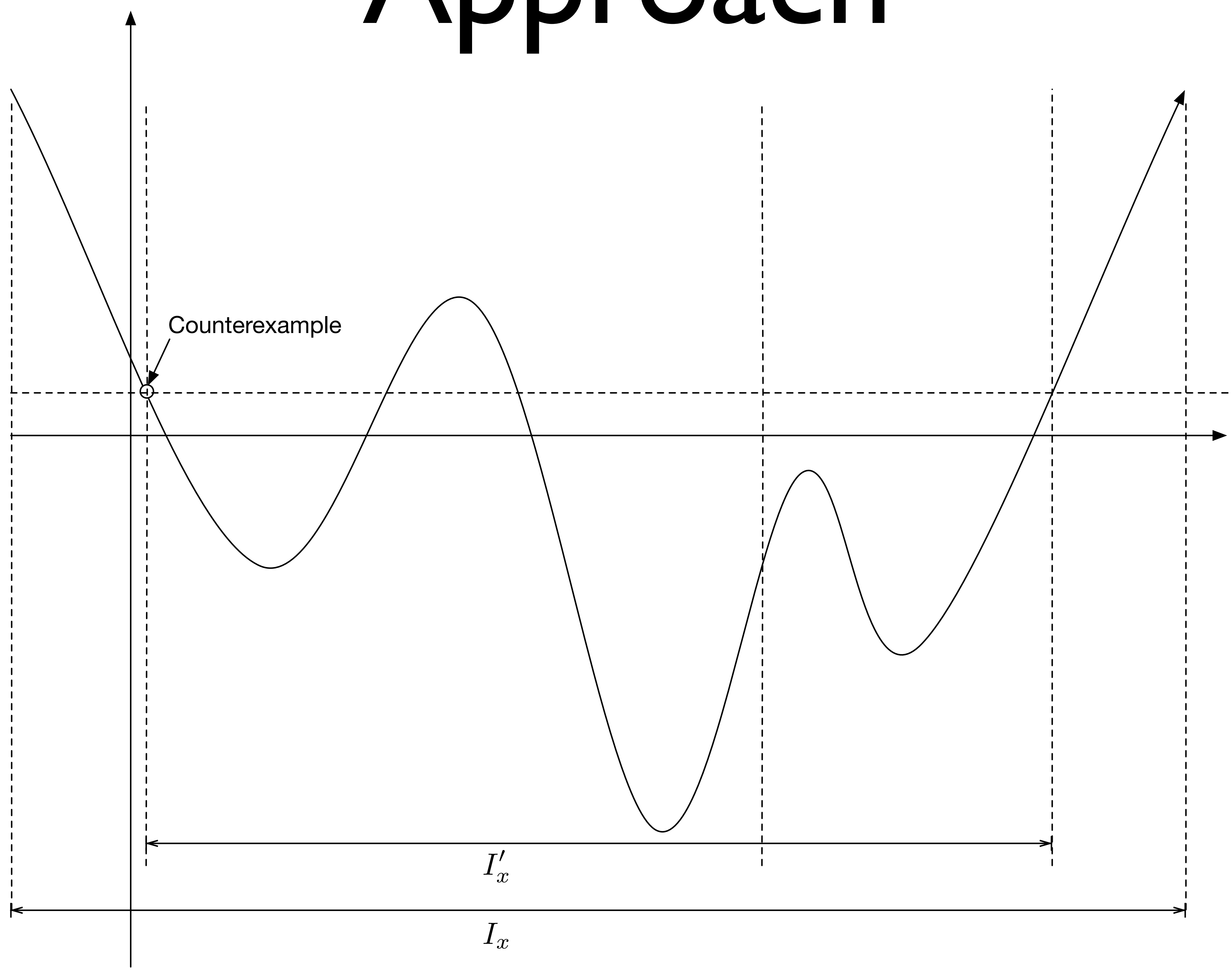
Solving Exist-forall Formulas

Approach



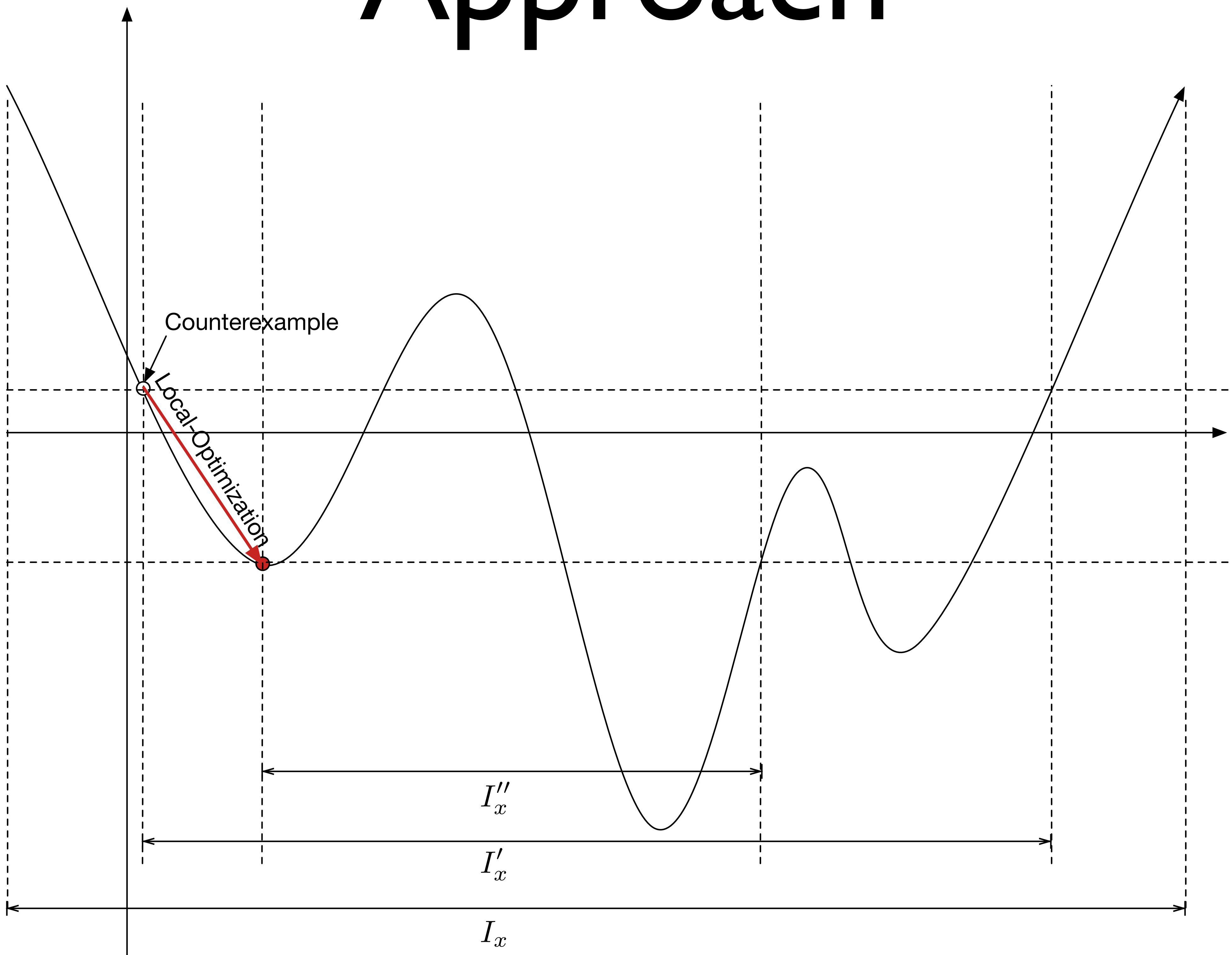
Solving Exist-forall Formulas

Approach



Solving Exist-forall Formulas

Approach



Solving Exist-forall Formulas

Approach

Exploit the **structure** of **optimization** problem:

$$\exists x.\forall y.f(x) \leq f(y)$$

1. Counterexample generation:

$$b \leftarrow \mathrm{Solve}(y, \neg\varphi(x, y))$$

2. Use **local-optimization** to **enhance the quality** of a **counterexample**:

$$b \leftarrow \mathrm{localOpt}(f, b)$$

3. Pruning on x using the counterexample b = y:

$$B_x \leftarrow \mathrm{Prune}(B_x, \neg\varphi(x, b))$$

Repeat until it fails to find a counterexample in step 1 or reaches a fixedpoint.

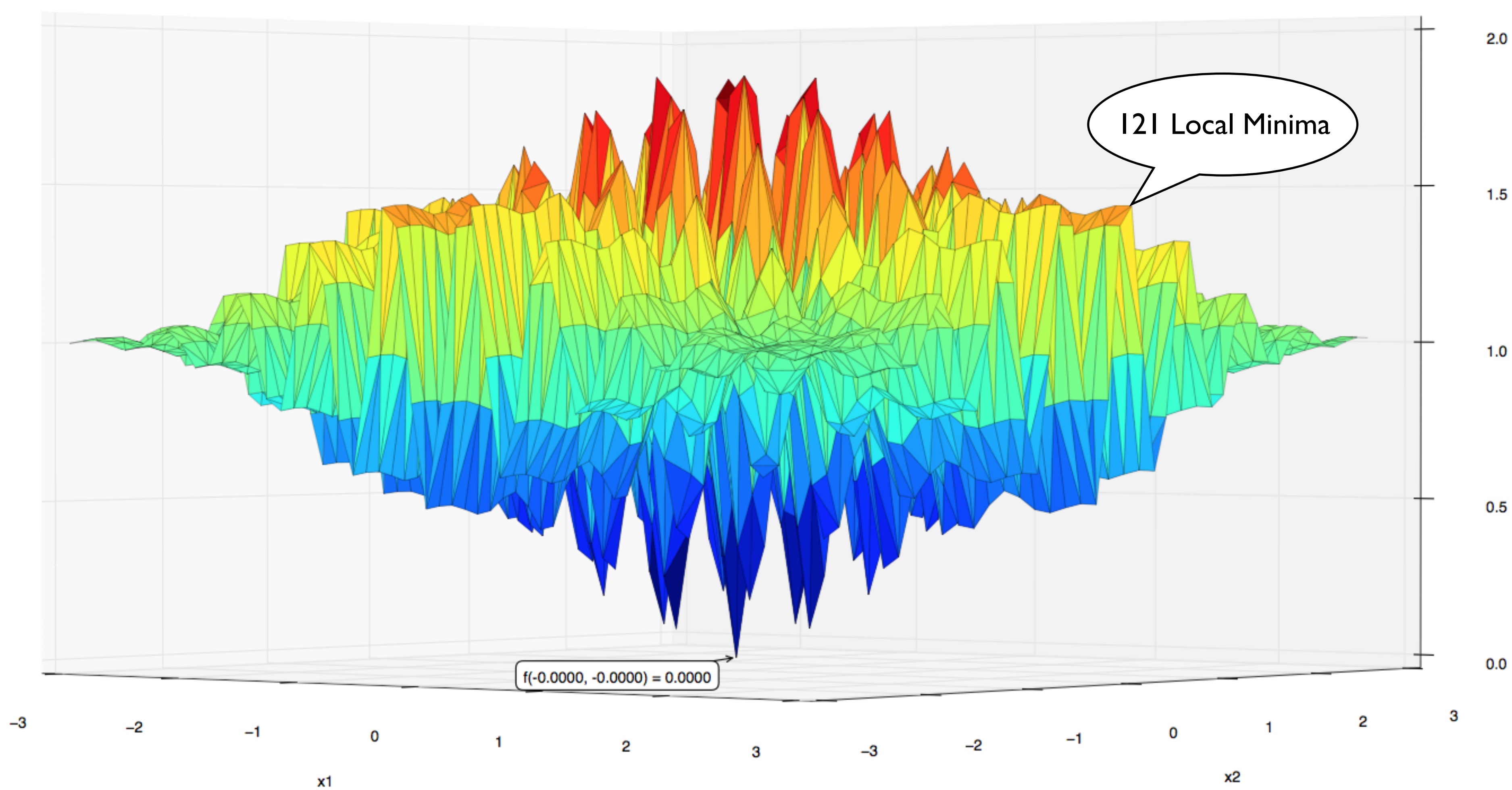
Solving Exist-forall Formulas

Preliminary Results

W / Wavy Function (#165 in [40])

$$f_{165}(x_1, x_2) = 1 - \frac{1}{2}\sum_{i=1}^{2} \cos(10 * x_i) e^{\frac{-x_i^2}{2}}$$

subject to $-3 \leq x_1, x_2 \leq 3$.



Momin Jamil and Xin-She Yang, A literature survey of benchmark functions for global optimization problems, Int. Journal of Mathematical Modelling and Numerical Optimisation

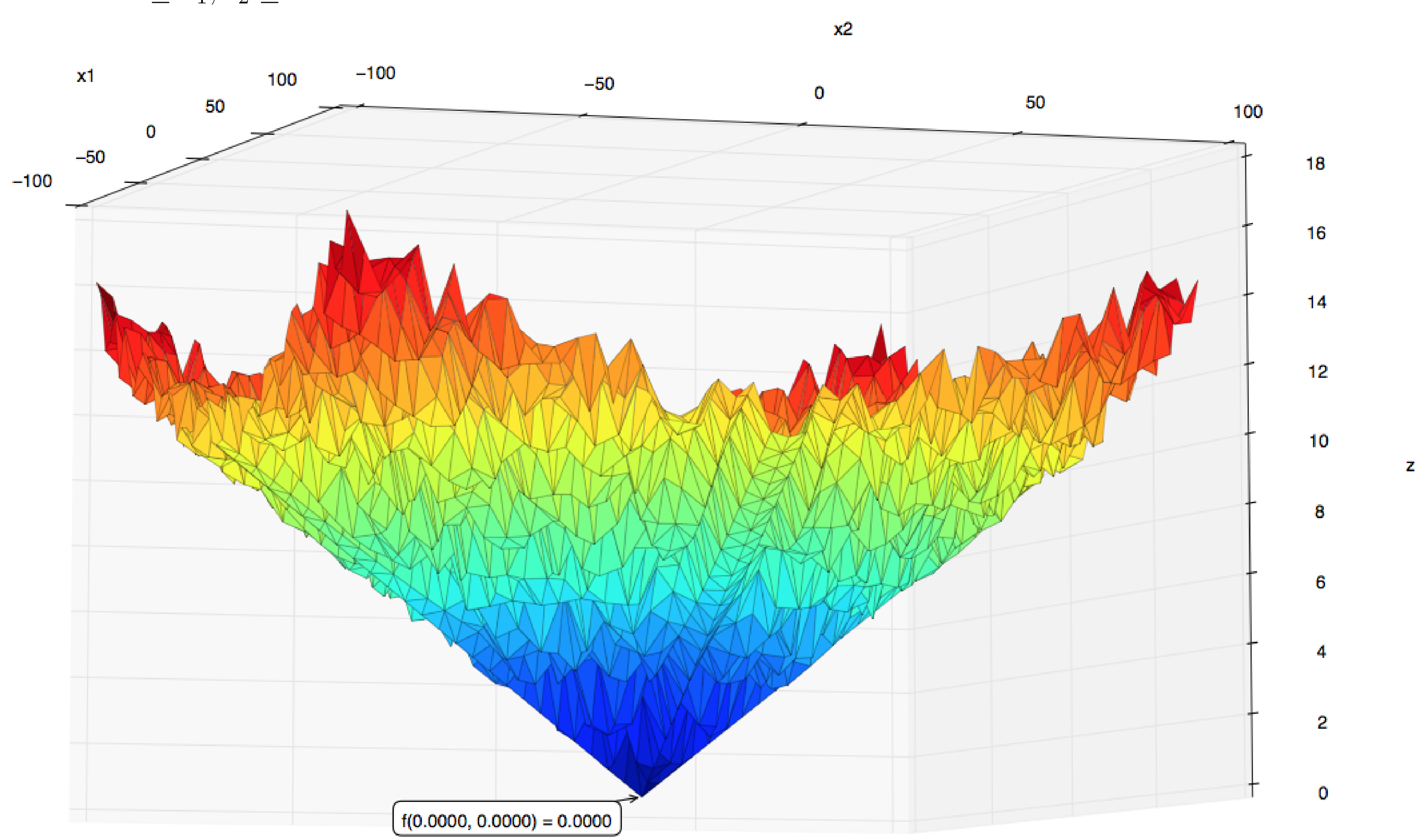
Solving Exist-forall Formulas

Preliminary Results

Salomon Function (#74 in [40])

$$f_{110}(x_1, x_2) = 1 - \cos\left(2\pi\sqrt{x_1^2 + x_2^2}\right) + 0.1\sqrt{x_1^2 + x_2^2}$$

subject to $-100 \le x_1, x_2 \le 100$.



Momin Jamil and Xin-She Yang, A literature survey of benchmark functions for global optimization problems, Int. Journal of Mathematical Modelling and Numerical Optimisation

Solving Exist-forall Formulas

Proposed Work

- Prove that the pruning algorithms **terminate**.
- Prove that the pruning algorithms are **well-defined**.
- Finish the **implementation**, run **experiments**.

Time Line &
Summary of Proposed Work

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Solving Exist-forall Formulas

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- Prove that the pruning algorithms are **well-defined**.
- Finish the **implementation**, run **experiments**.

Plan to finish all before the beginning of Fall semester 2016.

Thank you