Basic Concepts of Abstract Interpretation*

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*Work of P. Cousot and R.Cousot

P. Cousot and R. Cousot.
Basic Concepts of Abstract Interpretation.
In *Building the Information Society*, R. Jacquard (Ed.), pages 359–366. Kluwer Academic Publishers 2004.

Goal

To Understand basic concepts of abstract interpretation.

Contents

Introduction

Abstract Interpretation:
a theory of approximation of mathematical structures, in particular those involved in the semantic models of computer systems.

Transition Systems

Programs are formalized as transition systems τ:

$$\tau = \langle \Sigma, \Sigma_i, t \rangle$$

- Σ : a set of states
- $\Sigma_i \subseteq \Sigma$: the set of initial states
- $t \subseteq \Sigma \times \Sigma$: a transition relation between a state and its possible successors.

Example, the transition system

$$\langle \mathbb{Z}, \{0\}, \{\langle x, x' \rangle \mid x' = x + 1\} \rangle$$

of program $\mathtt{x} := 0$; $\mathtt{while\ true\ do\ x} := \mathtt{x} + 1$.

Partial Trace Semantics

A finite partial execution trace : $\sigma = s_0 s_1 \ldots s_n$

- $s_0 \in \Sigma$
- For all $i < n$, $\langle s_i, s_{i+1} \rangle \in t$

Partial traces of length 0 : ϕ
Partial traces of length 1 : $\Sigma_\tau^1 = \{s \mid s \in \Sigma\}$
Partial traces of length n + 1 :

$$\Sigma_\tau^{n+1} = \{\sigma s s' \mid \sigma s \in \Sigma_\tau^n \wedge \langle s, s' \rangle \in t\}$$

Collecting semantics of τ : all partial traces of all finite lengths

$$\Sigma_\tau^{\overrightarrow{*}} = \bigcup_{n \geqslant 0} \Sigma_\tau^n$$

Partial Trace Semantics in Fixpoint Form

For the function $\mathcal{F}_{\tau}^{\overrightarrow{*}}$

$$\mathcal{F}_{\tau}^{\overrightarrow{*}}(X) = \{s \mid s \in \Sigma\} \cup \{\sigma s s' \mid \sigma s \in X \land \langle s, s' \rangle \in t\}$$

$\Sigma_{\tau}^{\overrightarrow{*}}$ is the least fixpoint of $\mathcal{F}_{\tau}^{\overrightarrow{*}}$, that is

- $\mathcal{F}_{\tau}^{\overrightarrow{*}}(\Sigma_{\tau}^{\overrightarrow{*}}) = \Sigma_{\tau}^{\overrightarrow{*}}$
- For all X such that $\mathcal{F}_{\tau}^{\overrightarrow{*}}(X) = X$, $\Sigma_{\tau}^{\overrightarrow{*}} \subseteq X$

Therefore,

$$\Sigma_{\tau}^{\overrightarrow{*}} = \mathtt{lfp}\, \mathcal{F}_{\tau}^{\overrightarrow{*}} = \bigcup_{n \geqslant 0} \mathcal{F}_{\tau}^{\overrightarrow{*}\, n}(\phi)$$

Partial Trace Semantics in Fixpoint Form - Proof I

$\mathcal{F}^{\overrightarrow{*}}_{\tau}(\Sigma^{\overrightarrow{*}}_{\tau}) = \Sigma^{\overrightarrow{*}}_{\tau}$

The proof is as follows:

$$\mathcal{F}^{\overrightarrow{*}}_{\tau}(\Sigma^{\overrightarrow{*}}_{\tau}) = \mathcal{F}^{\overrightarrow{*}}_{\tau}(\bigcup_{n \geqslant 0} \Sigma^{n}_{\tau}) \qquad \text{def.} \Sigma^{\overrightarrow{*}}_{\tau}$$

$$= \{s \mid s \in \Sigma\} \cup \{\sigma s s' \mid \sigma s \in (\bigcup_{n \geqslant 0} \Sigma^{n}_{\tau}) \wedge \langle s, s' \rangle \in t\} \qquad \text{def. } \mathcal{F}^{\overrightarrow{*}}_{\tau}$$

$$= \{s \mid s \in \Sigma\} \cup \bigcup_{n \geqslant 0} \{\sigma s s' \mid \sigma s \in (\Sigma^{n}_{\tau}) \wedge \langle s, s' \rangle \in t\} \qquad \text{set theory}$$

$$= \Sigma^{1}_{\tau} \cup \bigcup_{n \geqslant 0} \Sigma^{n+1}_{\tau} \qquad \text{def. } \Sigma^{1}_{\tau} \text{ and } \Sigma^{n+1}_{\tau}$$

$$= \bigcup_{n' \geqslant 1} \Sigma^{n'}_{\tau} = \bigcup_{n \geqslant 0} \Sigma^{n}_{\tau}$$

by letting $n' = n + 1$ and since $\Sigma^{n}_{\tau} = \phi$

Partial Trace Semantics in Fixpoint Form - Proof II

For all X such that $\mathcal{F}_{\tau}^{\overrightarrow{*}}(X) = X$, $\Sigma_{\tau}^{\overrightarrow{*}} \subseteq X$
We prove by induction that $\forall n \geqslant 0 : \Sigma_{\tau}^{n} \subseteq X$

1. Base Case : $\Sigma_{\tau}^{0} = \phi \subseteq X$
2. Inductive Hypothesis : $\Sigma_{\tau}^{n} \subseteq X$
   Since $\sigma s \in \Sigma_{\tau}^{n} \to \sigma s \in X$,
   $\{\sigma s s' \mid \sigma s \in \Sigma_{\tau}^{n} \wedge \langle s, s' \rangle \in t\} \subseteq \{\sigma s s' \mid \sigma s \in X \wedge \langle s, s' \rangle \in t\}$
   Therefore,
   $$\Sigma_{\tau}^{n+1} \subseteq \mathcal{F}_{\tau}^{\overrightarrow{*}}(\Sigma_{\tau}^{n}) \subseteq \mathcal{F}_{\tau}^{\overrightarrow{*}}(X) = X$$

The Reflexive Transitive Closure Semantics as an Abstraction

- Abstraction of the partial trace semantics

$$\alpha^*(X) = \{\overrightarrow{\alpha}(\sigma) \mid \sigma \in X\} \qquad \text{where } \overrightarrow{\alpha}(s_0 s_1 \ldots s_n) = \langle s_0, s_n \rangle$$

  $\alpha^*(\Sigma_\tau^{\overrightarrow{*}})$ is the reflexive transitive closure t^* of the transition relation t.

- Concretization

$$\gamma^*(Y) = \{\sigma \mid \overrightarrow{\alpha}(\sigma) \in Y\} = \{s_0 s_1 \ldots s_n \mid \langle s_0, s_n \rangle \in Y\}$$

- $X \subseteq \gamma^*(\alpha^*(X))$

Answering Concrete Questions in the Abstract

Answering concrete question about X using a simpler abstract question on $\alpha^*(X)$.

Example : $s \ldots s' \ldots s'' \in X? \rightarrow \langle s, s'' \rangle \in \alpha^*(X)?$

Galois Connections

Given any set X of partial traces and Y of pair of states,

$$\alpha^*(X) \subseteq Y \iff X \subseteq \gamma^*(Y)$$

which is a characteristic property of Galois connections.
Proof.

$$\begin{aligned}
\alpha^*(X) \subseteq Y &\iff \{\overrightarrow{\alpha}^*(\sigma) \mid \sigma \in X\} \subseteq Y && \text{def. } \alpha^* \\
&\iff \forall \sigma \in X : \overrightarrow{\alpha}(\sigma) \in Y \\
&\iff X \subseteq \{\sigma \mid \overrightarrow{\alpha}(\sigma) \in Y\} && \text{def. } \subseteq \\
&\iff X \subseteq \gamma^*(Y) && \text{def. } \gamma^*
\end{aligned}$$

Galois Connections

Galois connections preserve joins.

$$\alpha^*(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \alpha^*(X_i)$$

Proof.

$$\begin{aligned}
\alpha^*(\bigcup_{i \in I} X_i) &= \{\overrightarrow{\alpha}^*(\sigma) \mid \sigma \in \bigcup_{i \in I} X_i\} \\
&= \bigcup_{i \in I} \{\overrightarrow{\alpha}^*(\sigma) \mid \sigma \in X_i\} \\
&= \bigcup_{i \in I} \alpha^*(X_i)
\end{aligned}$$

The Reflexive Transitive Closure Semantics in Fixpoint Form

* General Principle in Abstract Interpretation.

1. The concrete(partial trace) semantics is expressed in fixpoint form.

$$\Sigma_{\tau}^{\overrightarrow{*}} = \mathtt{lfp}\, \mathcal{F}_{\tau}^{\overrightarrow{*}}$$

2. The abstract(reflexive transitive closure) semantics is an abstraction of the concrete semantics by a Galois connections and it can be expressed in fixpoint form, too.

$$\alpha^{*}(\Sigma_{\tau}^{\overrightarrow{*}}) = \mathtt{lfp}\, \mathcal{F}_{\tau}^{*}$$

3. 2 can be generalized to order theory, and is known as the fixpoint transfer theorem.

The Reflexive Transitive Closure Semantics in Fixpoint Form - Propositions & Definitions

1. Proposition 1. $\alpha^*(\phi) = \phi$
   $\phi \subseteq \gamma^*(\phi) \iff \alpha^*(\phi) \subseteq \phi$. Therefore $\alpha^*(\phi) = \phi$.
2. Propostion 2.
   Commutation Property: $\alpha^*(\mathcal{F}_\tau^{\overrightarrow{*}}(X)) = \mathcal{F}_\tau^*(\alpha^*(X))$
   1. Definition 1. $\mathbb{I}_\Sigma = \{\langle s, s\rangle \mid s \in \Sigma\}$
   2. Definition 2. $\mathcal{F}_\tau^*(Y) = \mathbb{I}_\Sigma \cup Y \circ t$

$$
\begin{aligned}
&\alpha^*(\mathcal{F}_\tau^{\overrightarrow{*}}(X)) \\
&= \alpha^*(\{s \mid s \in \Sigma\} \cup \{\sigma s s' \mid \sigma s \in X \wedge \langle s, s'\rangle \in t\}) && \text{def. } \mathcal{F}_\tau^{\overrightarrow{*}} \\
&= \{\overrightarrow{\alpha}(s) \mid s \in \Sigma\} \cup \{\overrightarrow{\alpha}(\sigma s s') \mid \sigma s \in X \wedge \langle s, s'\rangle \in t\}) && \text{def. } \alpha^* \\
&= \{\langle s, s\rangle \mid s \in \Sigma\} \cup \{\langle \sigma_0, s'\rangle \mid \exists s : \sigma s \in X \wedge \langle s, s'\rangle \in t\}) && \text{def. } \overrightarrow{\alpha} \\
&= \mathbb{I}_\Sigma \cup \{\langle \sigma_0, s'\rangle \mid \exists s : \langle \sigma_0, s\rangle \in \alpha^*(X) \wedge \langle s, s'\rangle \in t\}) && \text{def.} \mathbb{I}_\Sigma, \alpha^* \\
&= \mathbb{I}_\Sigma \cup \alpha^*(X) \circ t \\
&= \mathcal{F}_\tau^*(\alpha^*(X))
\end{aligned}
$$

The Reflexive Transitive Closure Semantics in Fixpoint Form - Proof

Showing

$$\alpha^*(\overrightarrow{\Sigma_\tau^*}) = \mathtt{lfp}\, \mathcal{F}_\tau^*$$

is equivalent to prove that

$$\alpha^*\Big(\bigcup_{n \geqslant 0} \overrightarrow{\mathcal{F}_\tau^*}^n(\phi)\Big) = \bigcup_{n \geqslant 0} {\mathcal{F}_\tau^*}^n(\phi)$$

Using induction on

$$\forall n : \alpha^*(\overrightarrow{\mathcal{F}_\tau^*}^n(\phi)) = {\mathcal{F}_\tau^*}^n(\phi)$$

The Reflexive Transitive Closure Semantics in Fixpoint Form - Proof

$$\forall n : \alpha^*(\overrightarrow{\mathcal{F}^*_\tau}^{\,n}(\phi)) = \mathcal{F}^{*\,n}_\tau(\phi)$$

1. Base Case:
   $\alpha^*(\overrightarrow{\mathcal{F}^*_\tau}^{\,0}(\phi)) = \phi = \mathcal{F}^{*\,0}_\tau(\phi)$
2. Inductive Hypothesis: $\alpha^*(\overrightarrow{\mathcal{F}^*_\tau}^{\,n}(\phi)) = \mathcal{F}^{*\,n}_\tau(\phi)$

$$\begin{aligned}
\alpha^*(\overrightarrow{\mathcal{F}^*_\tau}^{\,n+1}(\phi)) &= \alpha^*(\overrightarrow{\mathcal{F}^*_\tau}(\overrightarrow{\mathcal{F}^*_\tau}^{\,n}(\phi))) \\
&= \mathcal{F}^*_\tau(\alpha^*(\overrightarrow{\mathcal{F}^*_\tau}^{\,n}(\phi))) && \text{commutative} \\
&= \mathcal{F}^*_\tau \mathcal{F}^{*\,n}_\tau(\phi) && \text{inductive hypothesis} \\
&= \mathcal{F}^{*\,n+1}_\tau(\phi)
\end{aligned}$$

The Reachability Semantics as an Abstraction

The reachability semantics of the transition system $\tau = \langle \Sigma, \Sigma_i, t \rangle$

$$\{s' \mid \exists s \in \Sigma_i : \langle s, s' \rangle \in t^*\}$$

is the set of states that are reachable from the initial states Σ_i.

The Reachability Semantics as an Abstraction

Definition $\text{post}[r]Z$: The right-image of the set Z by relation r

$$\text{post}[r]Z = \{s' \mid \exists s \in Z : \langle s, s' \rangle \in r\}$$

The Reachability Semantics as an Abstraction

Abstraction of the reflexive transitive closure semantics Y is defined as

$$\begin{aligned}\alpha^{\bullet}(Y) &= \{s' \mid \exists s \in \Sigma_i : \langle s, s' \rangle \in Y\} \\ &= \mathrm{post}[Y]\Sigma_i\end{aligned}$$

Concretization of the reachability semantics Z is defined as

$$\gamma^{\bullet}(Z) = \{\langle s, s' \rangle \mid s \in \Sigma_i \implies s' \in Z\}$$

Galois Connection

We have the Galois Connection:

$$\alpha^\bullet(Y) \subseteq Z \iff Y \subseteq \gamma^\bullet(Z)$$

Proof.

$$\begin{aligned}
\alpha^\bullet(Y) \subseteq Z &\iff \{s' \mid \exists s \in \Sigma_i : \langle s, s' \rangle \in Y\} \subseteq Z && \text{def. } \alpha^\bullet \\
&\iff \forall s' : \forall s \in \Sigma_i : \langle s, s' \rangle \in Y \Longrightarrow s' \in Z && \text{def. } \subseteq \\
&\iff \forall \langle s, s' \rangle \in Y : s \in \Sigma_i \Longrightarrow s' \in Z\} && \text{def. } \Longrightarrow \\
&\iff Y \subseteq \{\langle s, s' \rangle \mid s \in \Sigma_i \Longrightarrow s' \in Z\} && \text{def. } \subseteq \\
&\iff Y \subseteq \gamma^\bullet(Z) && \text{def. } \gamma^\bullet
\end{aligned}$$

The Reachability Semantics in fixpoint form

1. Define $\mathcal{F}^\bullet_\tau(Z) = \Sigma_i \cup \text{post}[t]Z$.
2. Establish commutation property $\alpha^\bullet(\mathcal{F}^*_\tau(Y)) = \alpha^\bullet(\mathcal{F}^\bullet_\tau(Y))$

$$
\begin{aligned}
& \alpha^\bullet(\mathcal{F}^*_\tau(Y)) & \\
= & \{s' \mid \exists s \in \Sigma_i : \langle s, s' \rangle \in (\mathbb{I}_\Sigma \cup Y \circ t)\} & \text{def. } \alpha^\bullet \& \mathcal{F}^*_\tau \\
= & \{s' \mid \exists s \in \Sigma_i : s' = s\} \cup & \\
& \{s' \mid \exists s \in \Sigma_i : \exists s'' : \langle s, s'' \rangle \in Y \wedge \langle s'', s' \rangle \in t\} & \text{def. } \mathbb{I}_\Sigma \& \circ \\
= & \Sigma_i \cup \{s' \mid \exists s'' \in \alpha^\bullet(Y) \wedge \langle s'', s' \rangle \in t\} & \text{def. } \alpha^\bullet \\
= & \alpha^\bullet(\mathcal{F}^\bullet_\tau(Y)) & \text{def } \mathcal{F}^\bullet_\tau(Z)
\end{aligned}
$$

3. By the fixpoint transfer theorem,

$$\alpha^\bullet(t^*) = \alpha^\bullet(\texttt{lfp}\, \mathcal{F}^*_\tau) = \texttt{lfp}\, \mathcal{F}^\bullet_\tau$$

The Interval Semantics as an Abstraction

The set of states of a transtion system $\tau = \langle \Sigma, \Sigma_i, t \rangle$ is totally ordered $\langle \Sigma, < \rangle$ with extrema $-\infty$ and $+\infty$, the interval semantics $\alpha^{\vdash\!\dashv}(\alpha^\bullet(t^*))$ of τ provides bounds on its reachable states $\alpha^\bullet(t*)$:

$$\alpha^{\vdash\!\dashv}(Z) = [\min Z, \max Z]$$

$$\min(\phi) = \infty \qquad \max(\phi) = -\infty$$

Concretization:

$$\gamma^{H}([l, h]) = \{s \in \Sigma \mid l \leqslant s \leqslant h\}$$

Abstract implication:

$$[l, h] \sqsubseteq [l', h'] \Longleftrightarrow (l' \leqslant l \wedge h \leqslant h')$$

Galois Connection

- We have the Galois Connection:

$$\alpha^{\vdash\!\dashv}(Z) \sqsubseteq [l, h] \iff Z \subseteq \gamma^{H}([l, h])$$

Proof.

$$\begin{aligned}
\alpha^{\vdash\!\dashv}(Z) \sqsubseteq [l, h] &\iff [\min Z, \max Z] \sqsubseteq [l, h] && \text{def. } \alpha^{\vdash\!\dashv} \\
&\iff l \leqslant \min Z \wedge \max Z \leqslant h && \text{def. } \sqsubseteq \\
&\iff Z \subseteq \{s \in \Sigma \mid l \leqslant s \leqslant h\} && \text{def. } \min\&\max \\
&\iff Z \subseteq \gamma^{H}([l, h]) && \text{def. } \gamma^{H}
\end{aligned}$$

- By defining

$$\bigsqcup_{i \in I} [l_i, h_i] = [\min_{i \in I} l_i, \max_{i \in I} h_i]$$

, Galois connection preserves least upper bounds

$$\alpha^{\vdash\!\dashv}(\bigcup_{i \in I} Z_i) = \bigsqcup_{i \in I} (Z_i)$$

The Interval Semantics in Fixpoint Form

1. Define $[\min \Sigma_i, \max \Sigma_i] \cup \alpha^{\vdash\!\dashv} \circ \mathrm{post}[t] \circ \gamma^{H}(I) \sqsubseteq \mathcal{F}^{H}_{\tau}(I)$
2. Establish semi-commutation property

$$\alpha^{\vdash\!\dashv}(\mathcal{F}^{\bullet}_{\tau}(Z)) \sqsubseteq \mathcal{F}^{H}_{\tau}(\alpha^{\vdash\!\dashv}(Z))$$

$$\begin{aligned}
\alpha^{\vdash\!\dashv}(\mathcal{F}^{\bullet}_{\tau}(Z)) &= \alpha^{\vdash\!\dashv}(\Sigma_i \cup \mathrm{post}[t]Z) && \text{def } \mathcal{F}^{\bullet}_{\tau} \\
&= \alpha^{\vdash\!\dashv}(\Sigma_i) \cup \alpha^{\vdash\!\dashv}(\mathrm{post}[t][Z]) && \text{Galois Connection} \\
&\sqsubseteq [\min \Sigma_i, \max \Sigma_i] \cup \alpha^{\vdash\!\dashv}(\mathrm{post}[t](\gamma^{H}(\alpha^{\vdash\!\dashv}(Z)))) \\
&\sqsubseteq \mathcal{F}^{H}_{\tau}(\alpha^{\vdash\!\dashv}(Z))
\end{aligned}$$

3. By the fixpoint approximation:

$$\alpha^{\vdash\!\dashv}(\mathcal{F}^{\bullet}_{\tau}(t^*)) = \alpha^{\vdash\!\dashv}(\mathtt{lfp}\, \mathcal{F}^{\bullet}_{\tau}) \sqsubseteq \mathtt{lfp}\, \mathcal{F}^{H}_{\tau}$$

Convergence Acceleration

In general, $\texttt{lfp}\,\mathcal{F}^{H}_{\tau} = \bigsqcup_{n \geqslant 0} \mathcal{F}^{H}_{\tau}(\phi = [+\infty, -\infty])$ diverge. Example, the transition system

$$\langle \mathbb{Z}, \{0\}, \{\langle x, x' \rangle \mid x' = x + 1\} \rangle$$

of program `x` := 0; `while true do x` := `x` + 1.

$$\mathcal{F}^{H}_{\tau}([l, h]) = [0, 0] \cup [l + 1, h + 1]$$

It diverges: $[+\infty, -\infty]$, $[0, 0]$, $[0, 1]$, $[0, 2]$, ...

Widening

To accelerate convergence, introduce a widening $\triangledown$ such that,

$$(X \sqsubseteq X \triangledown Y) \wedge (Y \sqsubseteq X \triangledown Y)$$

$$\begin{aligned} I^0 &= \phi = [+\infty, -\infty] & \\ I^{n+1} &= I^n & \text{if } \mathcal{F}_\tau^H(I^n) \sqsubseteq I^n \\ &= I^n \triangledown \mathcal{F}_\tau^H(I^n) & \mathit{otherwise}. \end{aligned}$$

limit I^λ is finite($\lambda \in \mathbb{N}$) and is a fixpoint overapproximation

$$\mathtt{lfp}\,\mathcal{F}_\tau^H \sqsubseteq I^\lambda$$

Example of Widening

An example of interval widening

1. choosing finite sequence

$$-\infty = r_0 < r_1 < \cdots < r_k = +\infty$$

2.

$$\begin{aligned} [+\infty, -\infty] \mathbin{\triangledown} [l, h] &= [l, h] \\ [l, h] \mathbin{\triangledown} [l', h'] &= [\text{if } l > l' \text{ then } max\{r_i | r_i \leqslant l'\} \text{ else } l, \\ &\qquad \text{if } h < h' \text{ then } min\{r_i | h' \leqslant r_i\} \text{ else } h] \end{aligned}$$

Example of Widening

Example, the transition system

$$\langle \mathbb{Z}, \{0\}, \{\langle x, x' \rangle \mid x' = x + 1\} \rangle$$

of program `x` $:=$ 0; `while x` $<$ 100 `do x` $:=$ `x` $+$ 1.

$$\mathcal{F}_{\tau}^{H}([l, h]) = [0, 0] \cup [l + 1, \min(99, h) + 1]$$

1. Sequence $r = -\infty < -1 < 0 < 1 < \infty$
2. $$\begin{aligned} I^0 &= [+\infty, -\infty] \\ I^1 &= [0, 0] \sqcup [1, 1] = [0, 1] \\ I^2 &= [0, 1] \sqcup [0, 2] = [0, +\infty] \\ I^3 &= [0, +\infty] \end{aligned}$$

Narrowing

The limit of an iteration with widening can be improved by a narrowing $\triangle$, such that

$$Y \sqsubseteq X \Longrightarrow Y \sqsubseteq (X \triangle Y) \sqsubseteq X$$

All terms in the iterates with narrowing

$$\begin{aligned} J^0 &= I^\lambda \\ J^{n+1} &= J^n \triangle \mathcal{F}_\tau^H(J^0) \end{aligned}$$

improve the result obtained by widening.

$$\texttt{lfp}\, \mathcal{F}_\tau^H \sqsubseteq J^n \sqsubseteq I^\lambda$$

Example of Narrowing

$$[l, h] \triangle [l', h'] = [\text{if } \exists i : l = r_i \text{ then } l' \text{ else } l,\ \text{if } \exists j : h = r_j \text{ then } h' \text{ else } h]$$

Example, the transition system

$$\langle \mathbb{Z}, \{0\}, \{\langle x, x' \rangle \mid x' = x + 1\} \rangle$$

of program `x` $:= 0$; `while x` < 100 `do x` $:=$ `x` $+ 1$.

$$J^0 = [0, +\infty]$$
$$J^1 = [0, +\infty] \triangle [0, 100] = [0, 100]$$
$$J^2 = [0, 100] \triangle [0, 100] = [0, 100]$$

Composition of Abstractions

The design of three abstractions of the partial trace semantics $\overrightarrow{\Sigma}^{*}_{\tau}$ of a transition system τ was compositional. Composition of Galois connections is a Galois connection so the successive arguments on sound approximation do compose nicely.

$$\alpha^{\vdash\!\dashv} \circ \alpha^{\bullet} \circ \alpha^{*}, \gamma^{*} \circ \gamma^{\bullet} \circ \gamma^{\mathrm{H}}$$

Hierarchy of Semantics

The four semantics of a transition system $\tau = \langle \Sigma, \Sigma_i, t \rangle$ form a hierarchy

1. Partial traces $\overrightarrow{\Sigma_\tau^*}$
2. Reflexive transitive closure $\alpha^*(\overrightarrow{\Sigma_\tau^*})$
3. Reachability $\alpha^\bullet \circ \alpha^*(\overrightarrow{\Sigma_\tau^*})$
4. Interval semantics $\alpha^{\vdash\!\dashv} \circ \alpha^\bullet \circ \alpha^*(\overrightarrow{\Sigma_\tau^*})$

Thanks

Thank you for listening.